Mental exercise: geometry of polynomials

Solve for x:

$$x^2 = 1$$
$$x^2 - 1 = 0$$
$$(x-1)(x+1) = 0$$
$$x = 1, -1$$

$$x^2 + 1 = 2x$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

$$x(x+8) = 4$$
$$x^2 + 8x = 4$$
$$x^2 + 8x - 4 = 0$$
$$x^2 + 8x + 16 - 16 - 4 = 0$$
$$(x+4)^2 - 16 - 4 = 0$$
$$(x+4)^2 - 20 = 0$$
$$(x+4)^2 = 20$$
$$(x+4) = \pm\sqrt{20}$$
$$7x^2 + 32x - 773 = 0$$
$$x = \pm\sqrt{20} - 4$$
$$x = \pm\sqrt{4 * 5} - 4$$
$$x = \pm 2\sqrt{5} - 4$$
$$x = \pm 2(\sqrt{5} - 2)$$

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x^2 = \pm\sqrt{-1}$$
$$x^2 = \pm i$$

$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3, 2$$

$$7x^2 + 32x - 773 = 0$$

Solve this general second-order polynomial:

$$ax^2 + bx + c = 0$$

$$\frac{(ax^2 + bx + c)}{a} = \frac{0}{a}$$

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}) - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}\frac{4a}{4a}}$$

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve this general second-order polynomial:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 parameters: a, b, c

a pair of solutions everywhere in XYZ

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$p = \frac{b}{a} \qquad q = \frac{c}{a}$$

$$x^2 + px + q = 0$$

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

2 parameters: p, q

a pair of solutions everywhere in XY

$x^2 + px + q = 0$

What do the general solutions look like?

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Case 1:

$$\sqrt{p^2 - 4q} = 0$$

$$x = \frac{-p}{2} \pm \frac{0}{2}$$

$$x = \frac{-p}{2}$$

Single solution, occuring when

$$\sqrt{p^2 - 4q} = 0$$

$$p^2 - 4q = 0$$

$p^2 = 4q$ can view as y=x^2

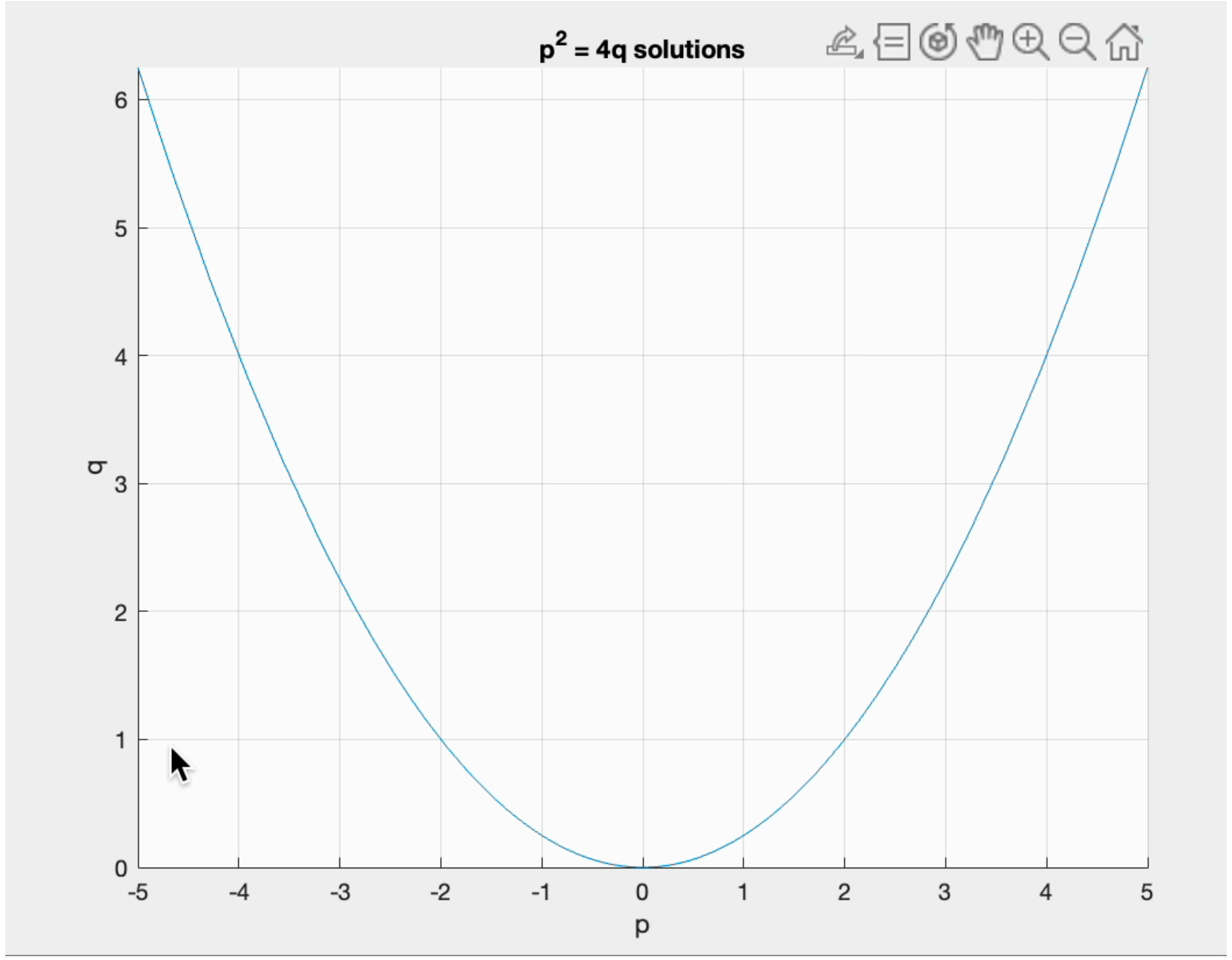
On that pq-plane parabola, solution is in vertical axis, with value z = -p/2

This is a tilted curve rising with negative p, falling with p

$x^2 + px + q = 0$ What do the general solutions look like?

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Case 1:



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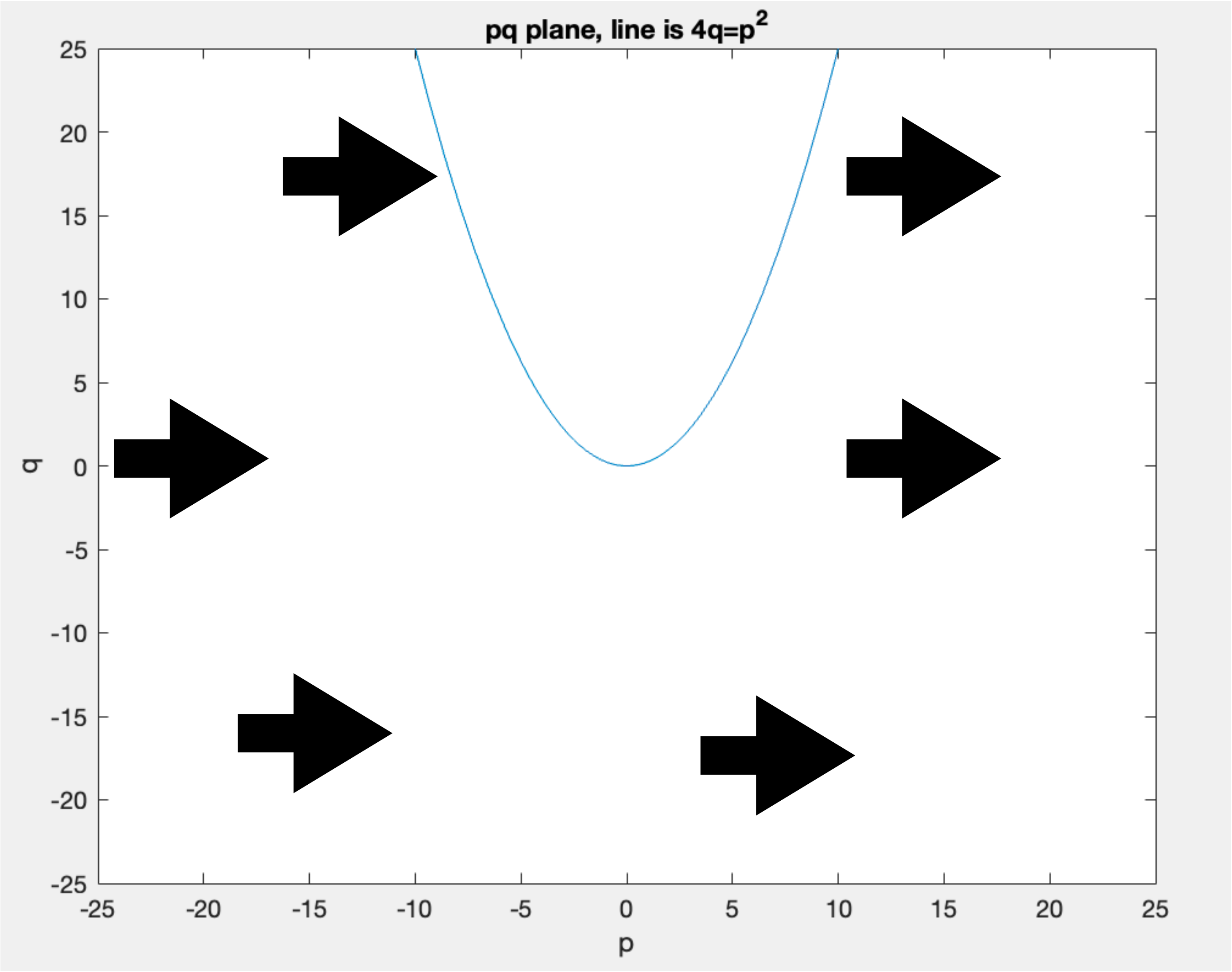


Case 2:

$$\sqrt{p^2 - 4q} > 0$$

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$$x = \frac{-p}{2} \pm \frac{\alpha}{2} \qquad \alpha \in \mathbb{R}$$

Solutions in z plane

Come in symmetric pairs about -p/2

Where alpha is linear in p

and sublinear in q

This is a curved, saddle-shaped surface

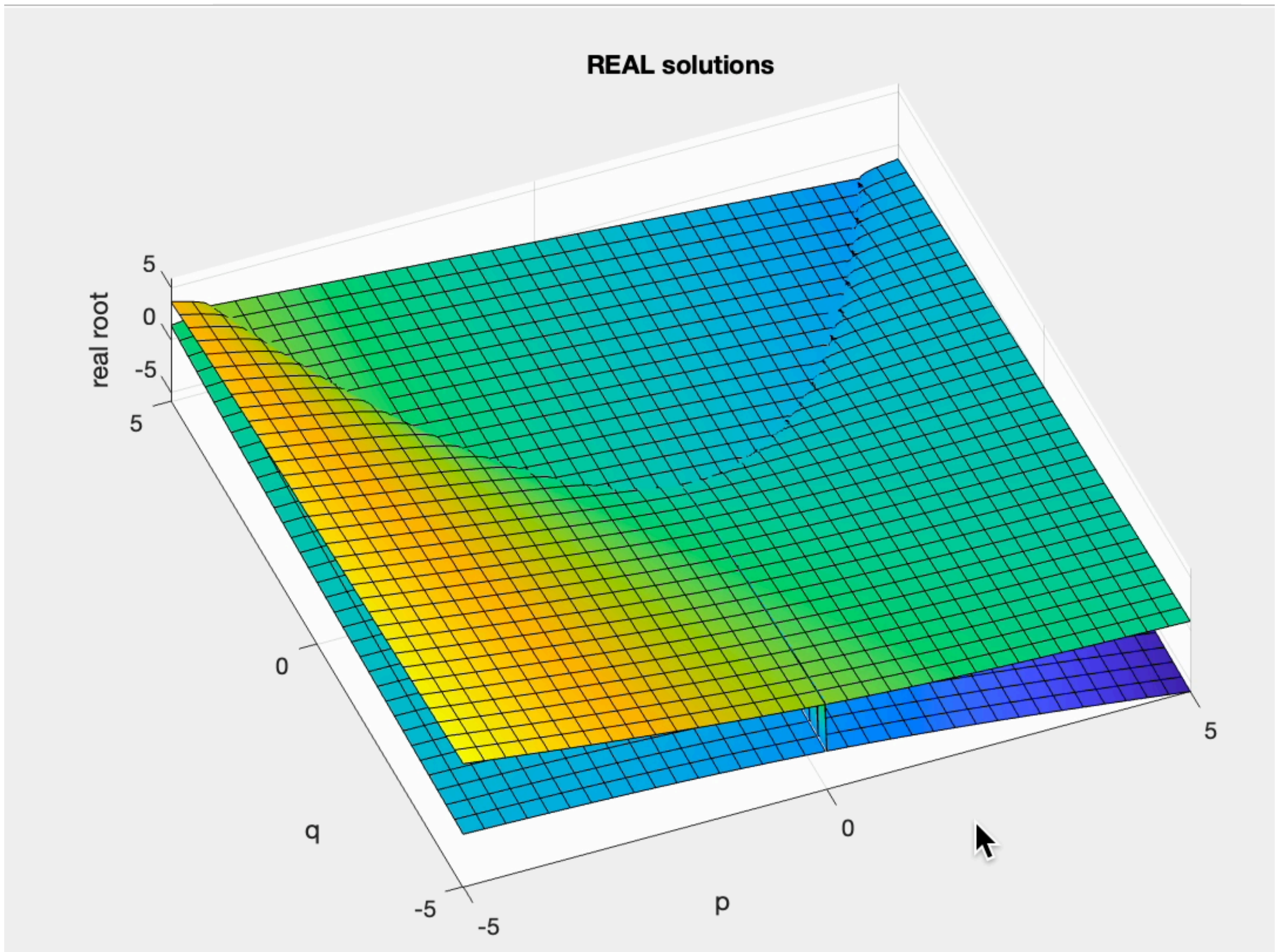
in the z dimension

tilted on z = -p/2

$x^2 + px + q = 0$

What do the general solutions look like?

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$



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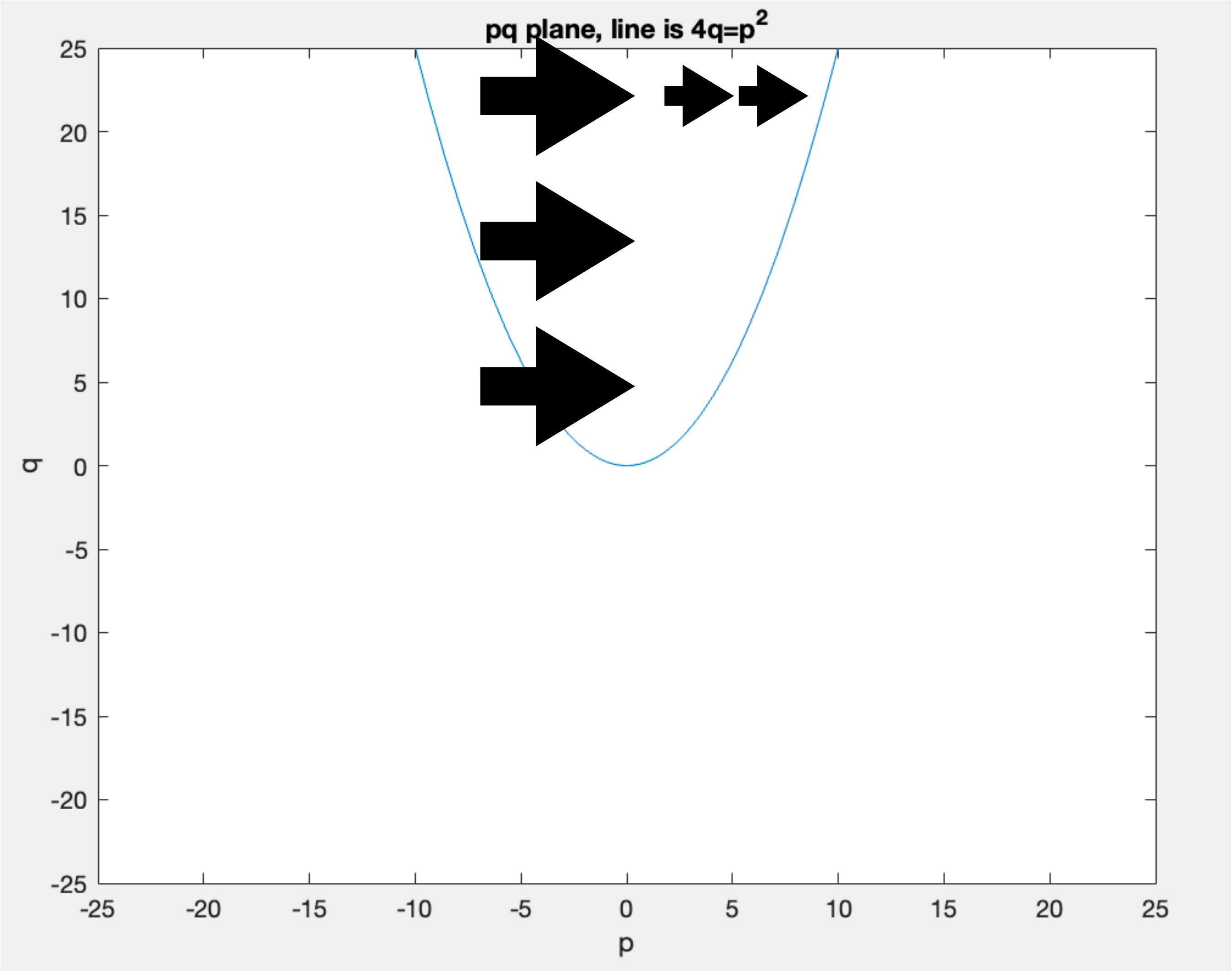


Case 3:

$$\sqrt{p^2 - 4q} < 0$$

solutions are complex numbers

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$$x = \frac{-p}{2} \pm \frac{\sqrt{-n}}{2}$$

$$x = \frac{-p}{2} \pm \alpha i$$

solutions have real component of -p/2 (same central element as other 2 cases)

and symmetric imaginary components

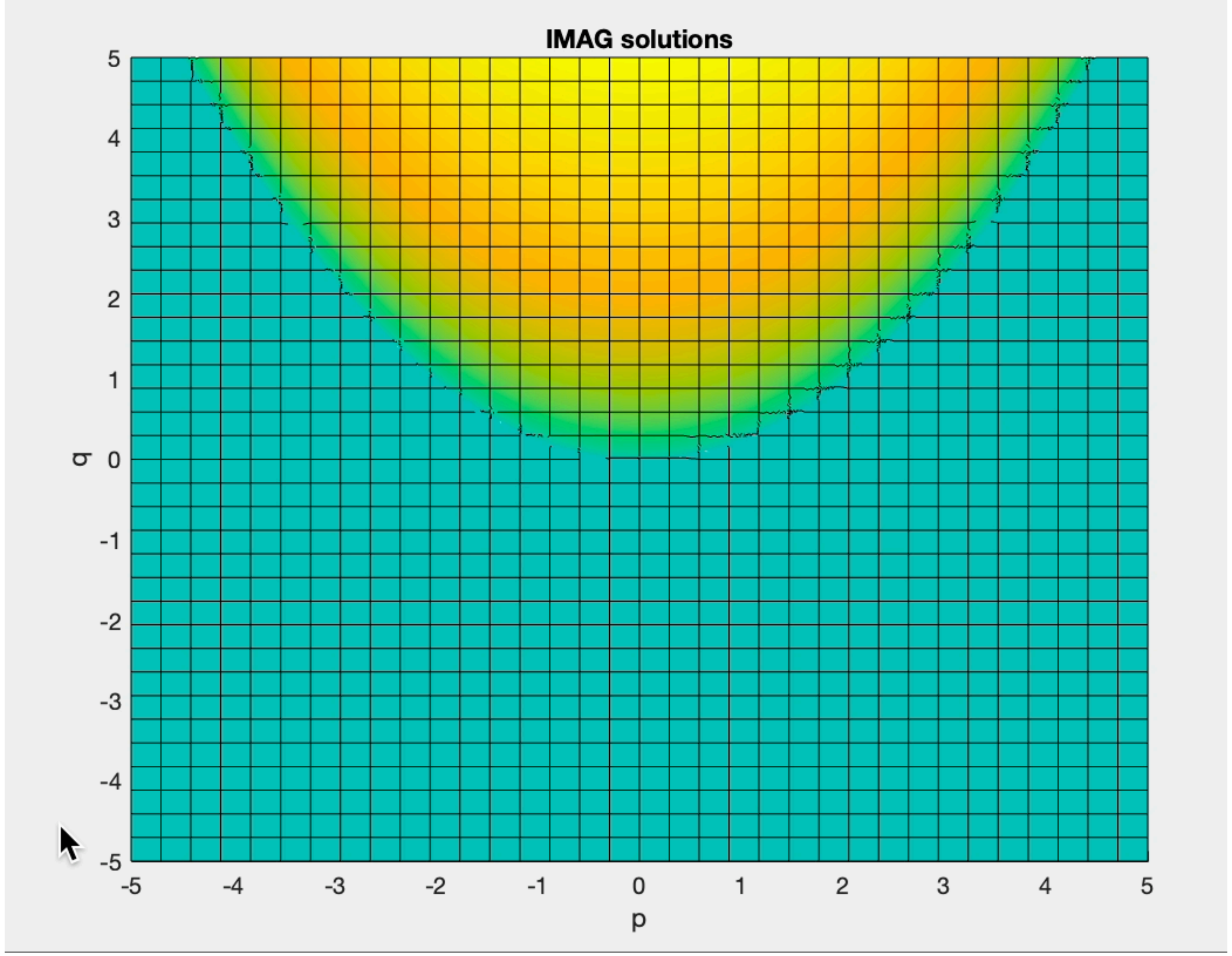
at fixed p, as q increases, imaginary magnitude increases

at fixed q, as p approaches 4q=p^2 parabola, imaginary magnitude goes to zero

this is 4D shape, but the imaginary part over pq is cup-shaped, fitting exactly in the saddle formed from cases 1 and 2

$x^2 + px + q = 0$

What do the general solutions look like?



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$x^2 + px + q = 0$

What do the general solutions look like?

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Case 3:

$\sqrt{p^2 - 4q} < 0$

Case 1:

$\sqrt{p^2 - 4q} = 0$

Case 2:

$\sqrt{p^2 - 4q} > 0$

We can't "see" the full solution space since it is 4D - we need a real pq plane, a real z, and an imaginary z

But we can add in the imaginary "as if" they were real to get a surrogate sense of the shape

$x^2 + px + q = 0$

What do the general solutions look like?

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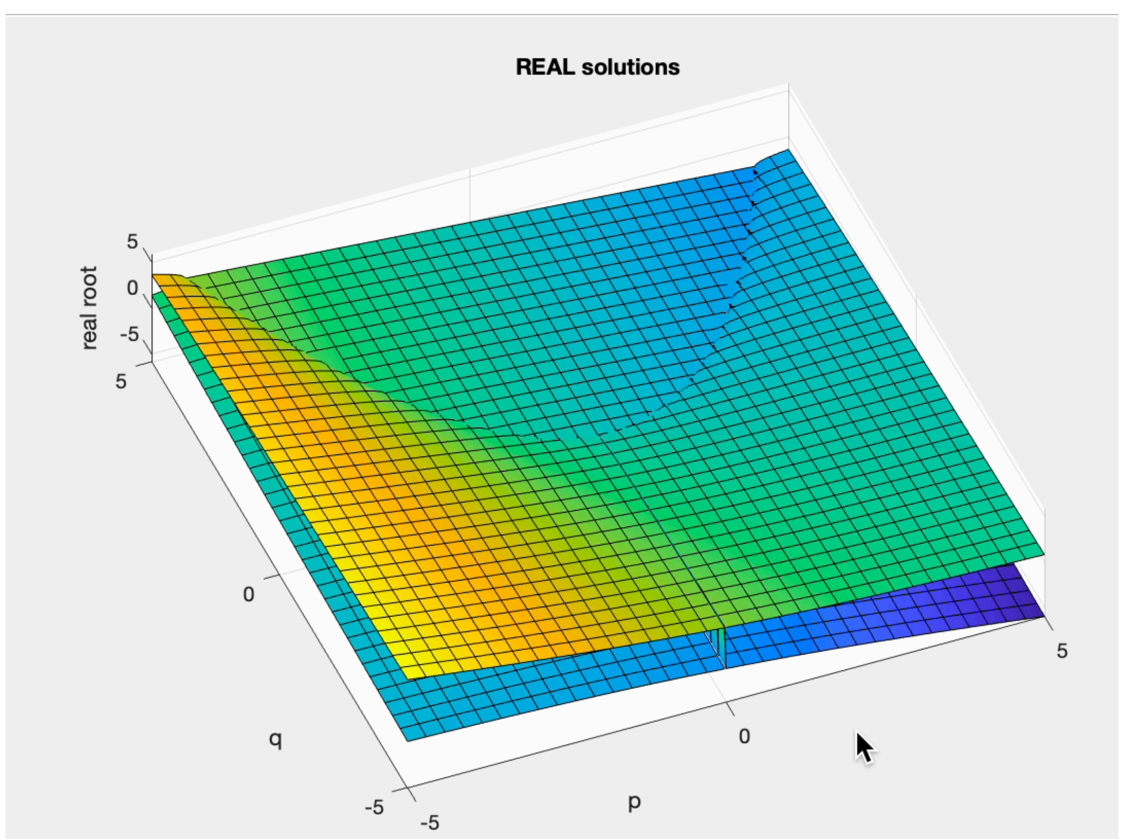
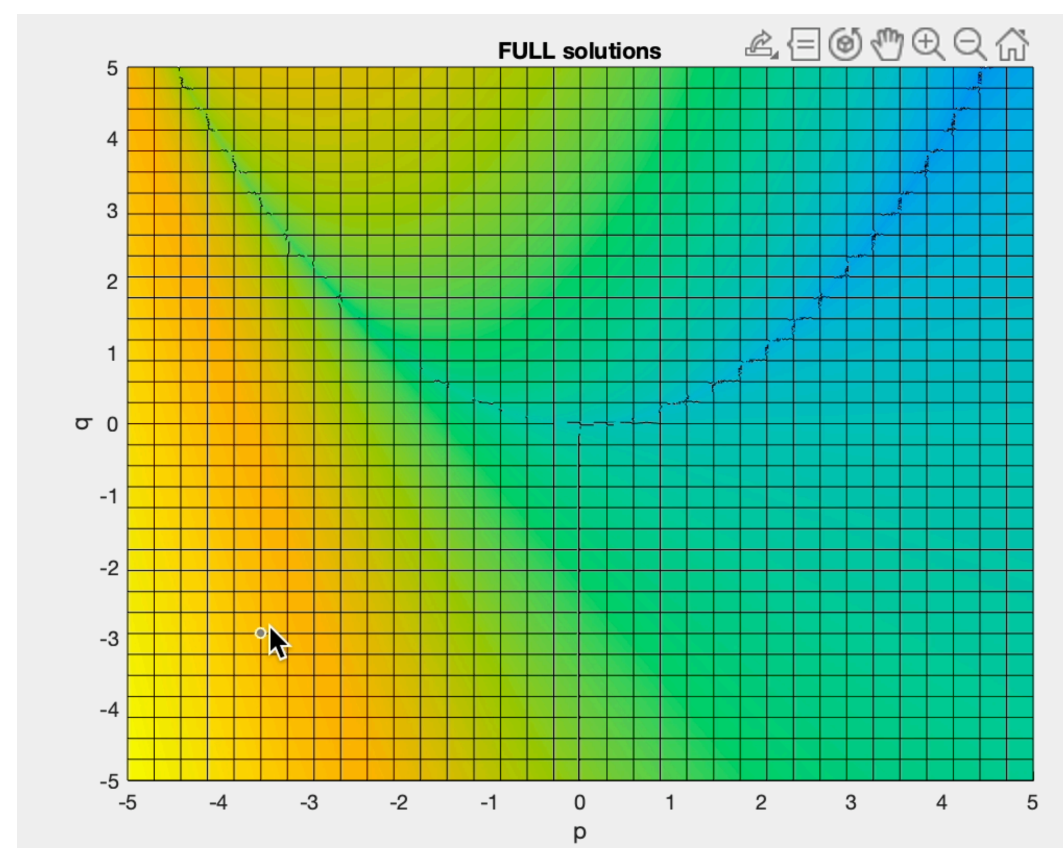




$x^2 + px + q = 0$

What do the general solutions look like?

$$x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

Case 3:

$\sqrt{p^2 - 4q} < 0$

$x_1, x_2 \in \mathbb{C}$

$x_1, x_2 = -\frac{p}{2} \pm Di$

Case 1:

$\sqrt{p^2 - 4q} = 0$

$x_1 \in \mathbb{R}$

$x_1 = -\frac{p}{2}$

Case 2:

$\sqrt{p^2 - 4q} > 0$

$x_1, x_2 \in \mathbb{R}$

$x_1, x_2 = -\frac{p}{2} \pm D$

Exact closed solution for any p,q - good!

Shape of solutions intelligible - good!

Straightforward (relatively) - good!

But… was it all even necessary?

Solve this general second-order polynomial:

$$ax^2 + bx + c = 0$$ 3 parameters

$$x^2 + px + q = 0$$ 2 parameters

Can we get down to 1 parameter?

$$y^2 + q' = 0$$ 1 parameter - could not be better

$$x = y + \delta$$

$$x^2 + px + q = 0$$

$$(y + \delta)^2 + p(y + \delta) + q = 0$$

$$y^2 + 2y\delta + \delta^2 + py + p\delta + q = 0$$

$$y^2 + y(2\delta + p) + (\delta^2 + p\delta + q) = 0$$

Write this form as

$$y^2 + yp' + q' = 0$$

We want p' = 0

Set 2d+p = 0

2d = -p   d = -p/2

$$\delta = -\frac{p}{2}$$

$$y^2 + y(2\delta + p) + (\delta^2 + p\delta + q) = 0$$

$$y^2 + y\left(2\left(\frac{-p}{2}\right) + p\right) + \left(\left(\frac{-p}{2}\right)^2 + p\left(\frac{-p}{2}\right) + q\right) = 0$$

$$y^2 + y(0) + \frac{p^2}{4} - \frac{p^2}{2} + q = 0$$

$$y^2 - \frac{p^2}{4} + q = 0$$

$$y^2 + \left(-\frac{p^2}{4} + q\right) = 0$$

$$y^2 + q' = 0 \qquad q' = q - \frac{p^2}{4}$$

Solve this general second-order polynomial:

$$ax^2 + bx + c = 0$$ 3 parameters

$$x^2 + px + q = 0$$ 2 parameters

Can we get down to 1 parameter?

$$y^2 + q' = 0$$ 1 parameter - could not be better

YES

$$x^2 + px + q = 0 \qquad x = \frac{-p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$$x = y - \frac{p}{2} \qquad q' = q - \frac{p^2}{4}$$

$$y^2 + 0y + q' = 0 \qquad y = \frac{0}{2} \pm \frac{\sqrt{0^2 - 4q'}}{2}$$

$$y = \pm \frac{\sqrt{-4q'}}{2}$$

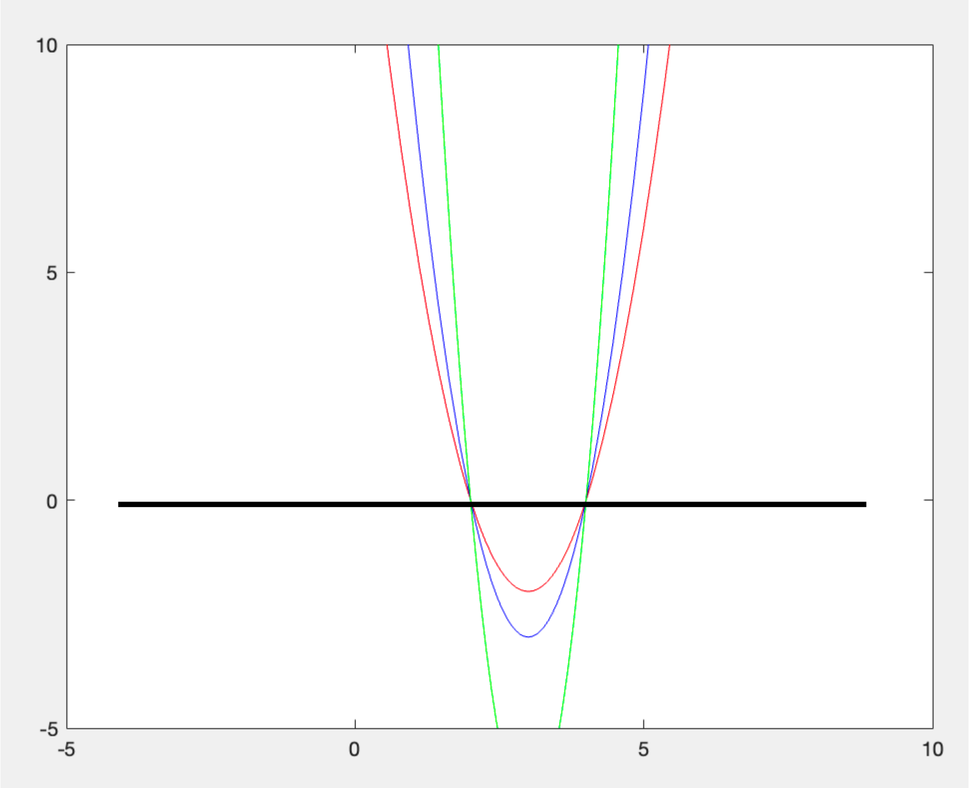
$$y = \pm \sqrt{-q'}$$

In sum, we converted an easy-to-understand second-order polynomial with parameters (a,b,c) to another second-order polynomial with just a single parameter (q')

What does the first step represent?

$$ax^2 + bx + c = 0$$

$$x^2 + px + q = 0$$



Note what happens here:

$$4x^2 - 12x + 8 = 0 \qquad (x^2 - 3x + 2) * 4 = 0$$

$$2x^2 - 6x + 4 = 0 \qquad (x^2 - 3x + 2) * 2 = 0$$

$$3x^2 - 9x + 6 = 0 \qquad (x^2 - 3x + 2) * 3 = 0$$

All reduce to:

$$x^2 - 3x + 2 = 0$$

Solutions are all identical

The (abc) parameter format contains redundancies: these are squeezed into a single case in the (pq) format.

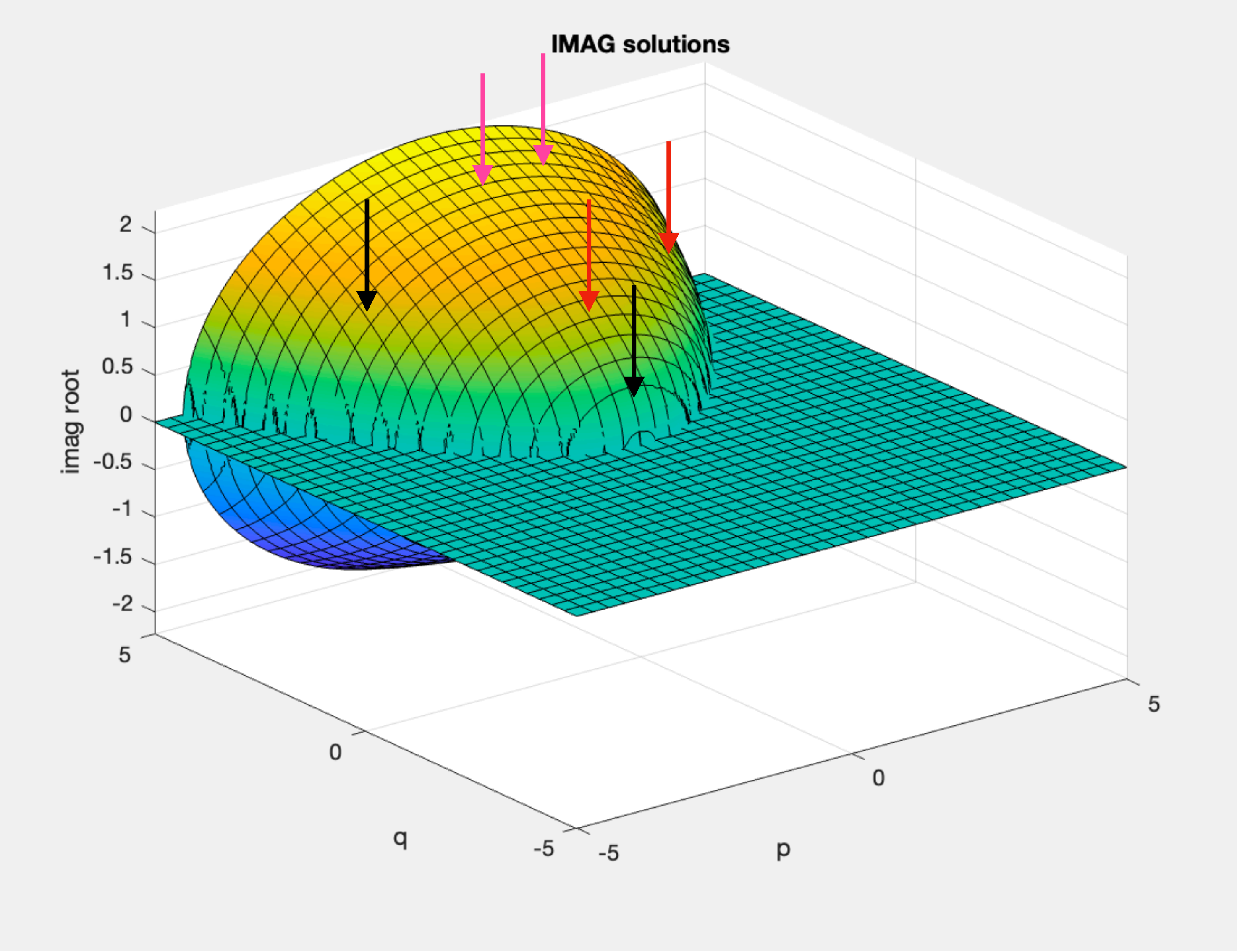
So there is never a reason to remain in (abc) format over (pq); it would be like using unreduced fractions.

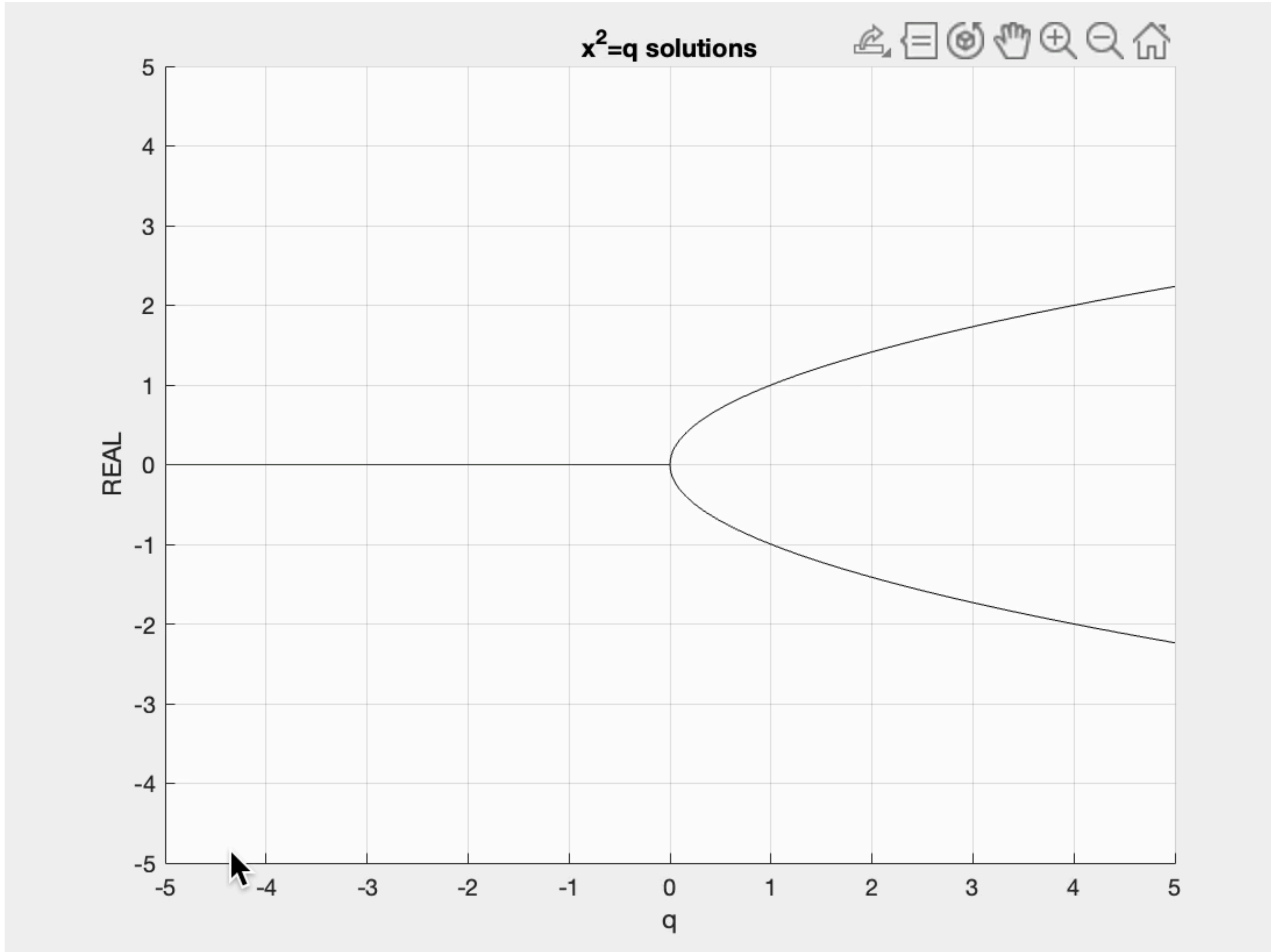
What does the second step represent?

$$x^2 + px + q = 0$$

$$y^2 + q' = 0 \qquad x = y - \frac{p}{2} \qquad q' = q - \frac{p^2}{4}$$

Note what happens here:





x²=q solutions
REAL
q
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

It is striking that all second order polynomials can be converted to a single square root calculation
(and that we are taught the quadratic equation, not the pq or q’ techniques)

But these results are fitting

A polynomial is governed by its highest power - it’s degree, here 2

We all know the parabola’s shape

Our steps here were to:
1) stretch/smush the parabola (abc to pq)
2) slide the parabola (pq to q’)

All this put a parabola of a normalized shape at the origin

These steps reduced our “solution space” from surfaces in 5D to a pair of lines in 3D

Q1: What is the geometric nature of the (pq) to (0q’) shift?

A family of pq points collapsed to a single q’ point on the p=0 coordinate. What is that family?

In reverse: what shape collapsed in (abc) to the point in (pq)? How did that happen?

Q2: Is it possible to eliminate terms of other polynomials?

In other words, can you make this shift occur?

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} \ldots = 0 \quad \Rightarrow \quad b_n x^n + 0 x^{n-1} + b_{n-2} x^{n-2} \ldots = 0$$

Try first to eliminate bx^2 in this cubic with x = y+w and choosing a useful w

$$ax^3 + bx^2 + cx + d = 0$$

Can you do it for a quartic, a quintic?

Can you write the general solution? The solutions for all b terms too?

Polynomials lacking a second term are called "depressed polynomials"
and they were the first route discovered to general closed form solutions of polynomial equations.

Sadly, closed solutions only exist for orders quadratic, cubic, and quartic; quintic and up have no closed solution.