Mental exercise: quadratics from other perspectives

We previously examined the solution geometry for any and all quadratics, proceeding through forms:

$$ax^2 + bx + c = 0$$

$$x^2 + px + q = 0$$

$$y^2 + q' = 0$$

Each form described any and all possible quadratic equations, but with fewer and fewer free parameters needed.

We obtained a surface (in 4D for pq form) or curve (in 3D for q' form) of the solutions.

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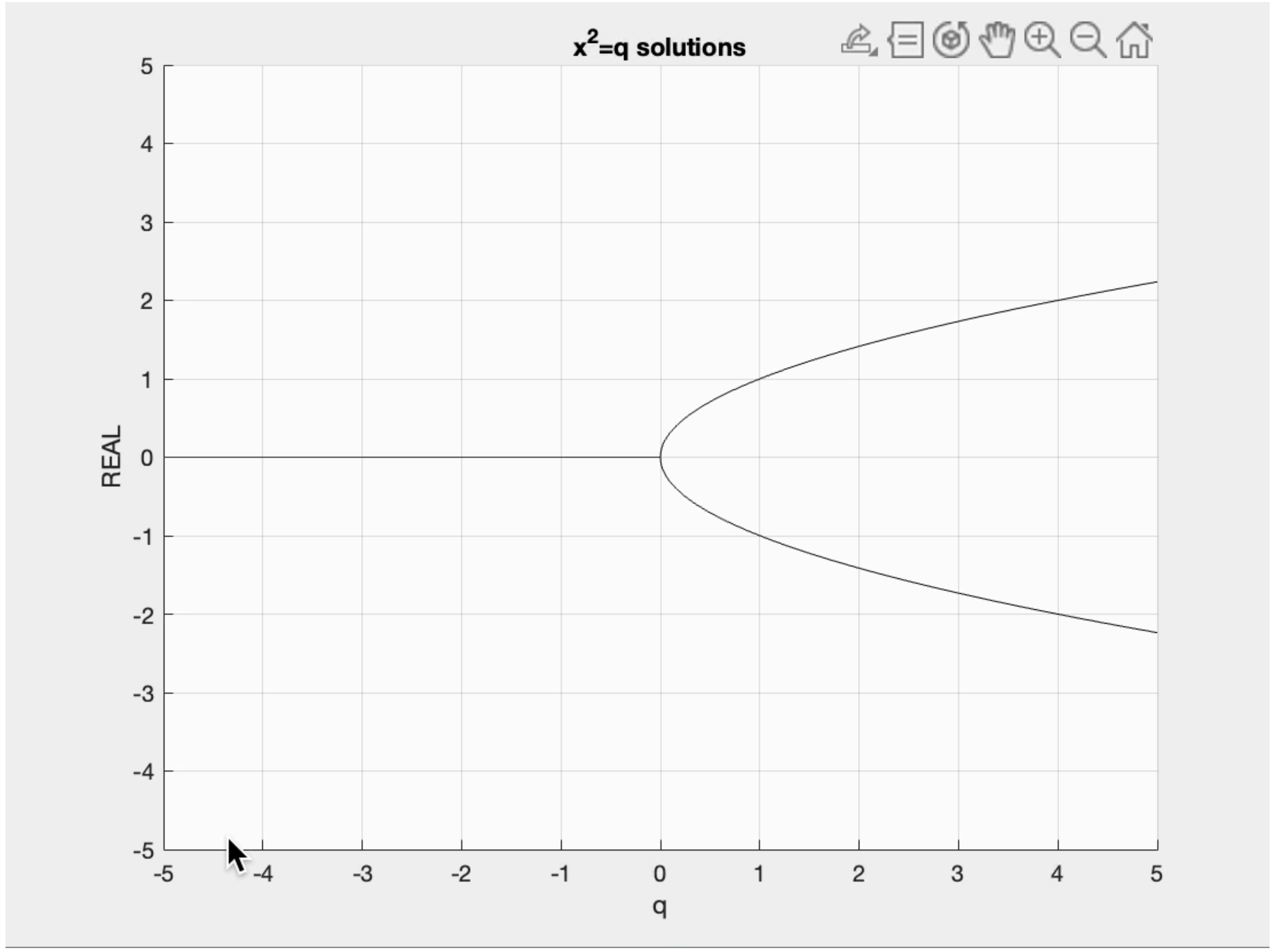

This was nice, but these quadratic formulas are only one way of seeing the situation. Let's try others.

Quadratics are fundamentally this:

$$(x-a)(x-b)=0$$

$$x^2-ax-bx+ab=0$$

$$\boxed{x^2-x(a+b)+ab=0} \qquad a+b=-p \qquad ab=q$$

$$x^2+px+q=0$$

Here is another way of writing any quadratic equation:

"Find two numbers whose sum is M, and whose product is N"

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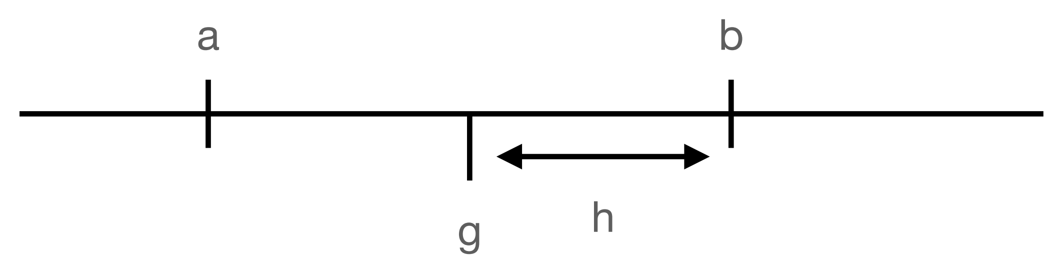
We know from the saddle shape of solutions that symmetry is fundamental to the solutions.

Pursue that idea. Can you reformulate the situation symmetrically?



$$a=g-h$$
$$b=g+h$$

$$g=\frac{b+a}{2}$$
$$h=\frac{b-a}{2}$$

$$a+b=-p$$

$$g=\frac{b+a}{2} \qquad g=-\frac{p}{2}$$

$$ab=q$$
$$(g-h)(g+h)=q$$
$$g^2-h^2=q$$

We know q, if only we knew g or h

Done! Look how nicely that fell out.

Fundamentally, we re-represented 2 constants as 2 new constants - the mean and variance - to get this simple solution

This is a versatile and often useful reframing for many kinds of problems

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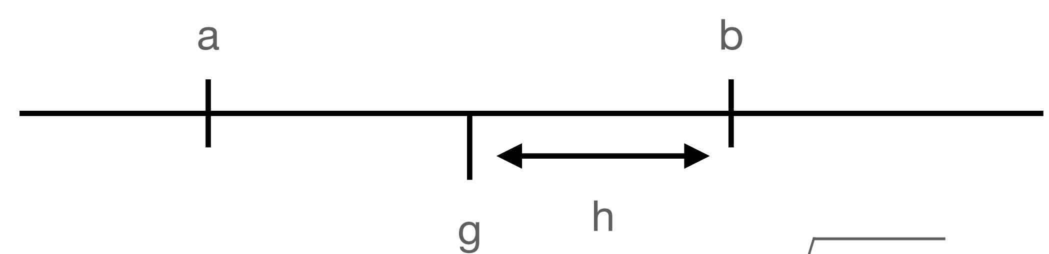

$$g=-\frac{p}{2} \qquad h=\frac{\sqrt{p^2-4q}}{2}$$

$$a=g-h$$

$$b=g+h$$

$$g=\frac{b+a}{2}$$

$$h=\frac{b-a}{2}$$

$$a+b=-p$$
$$ab=q$$

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$$g=\frac{b+a}{2} \qquad g=-\frac{p}{2}$$

$$ab=q$$
$$(g-h)(g+h)=q$$
$$g^2-h^2=q$$

We know q, if only we knew g or h

$$\frac{p^2}{4}-h^2=q$$

Done! Look how nicely that fell out.

$$h^2=\frac{p^2}{4}-q \qquad h=\sqrt{\frac{p^2}{4}-q} \qquad h=\frac{\sqrt{p^2-4q}}{2}$$

Connect this to the quadratic equation

$$a=g-h \qquad b=g+h$$

$$a=-\frac{p}{2}-\frac{\sqrt{p^2-4q}}{2} \qquad b=-\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}$$

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Another technique

$$a+b=-p$$

$$(a+b)^2=p^2$$

$$a^2+2ab+b^2=p^2$$

$$ab=q$$
$$-4ab=-4q$$

$$a^2-2ab+b^2=p^2-4q$$

$$(a-b)^2=p^2-4q$$

$$a-b=\sqrt{p^2-4q}$$

$$a+b=-p$$

$$2a=-p+\sqrt{p^2-4q}$$

$$2b=-p-\sqrt{p^2-4q}$$

Another versatile trick - make squares of sums into squares of differences by inverting the cross-term

Then you get conjugates to add and subtract for solutions

Where are these random solutions coming from?

Disclaimer: in next section, we presume a, b are positive real numbers so that geometry works

$$(x-a)(x-b)=0$$

$$x^2-x(a+b)+ab=0$$

$$x^2+px+q=0 \qquad a+b=-p \qquad ab=q$$

In other words, we are requiring p < 0 so that a+b > 0

and q >= 0 so that ab >= 0

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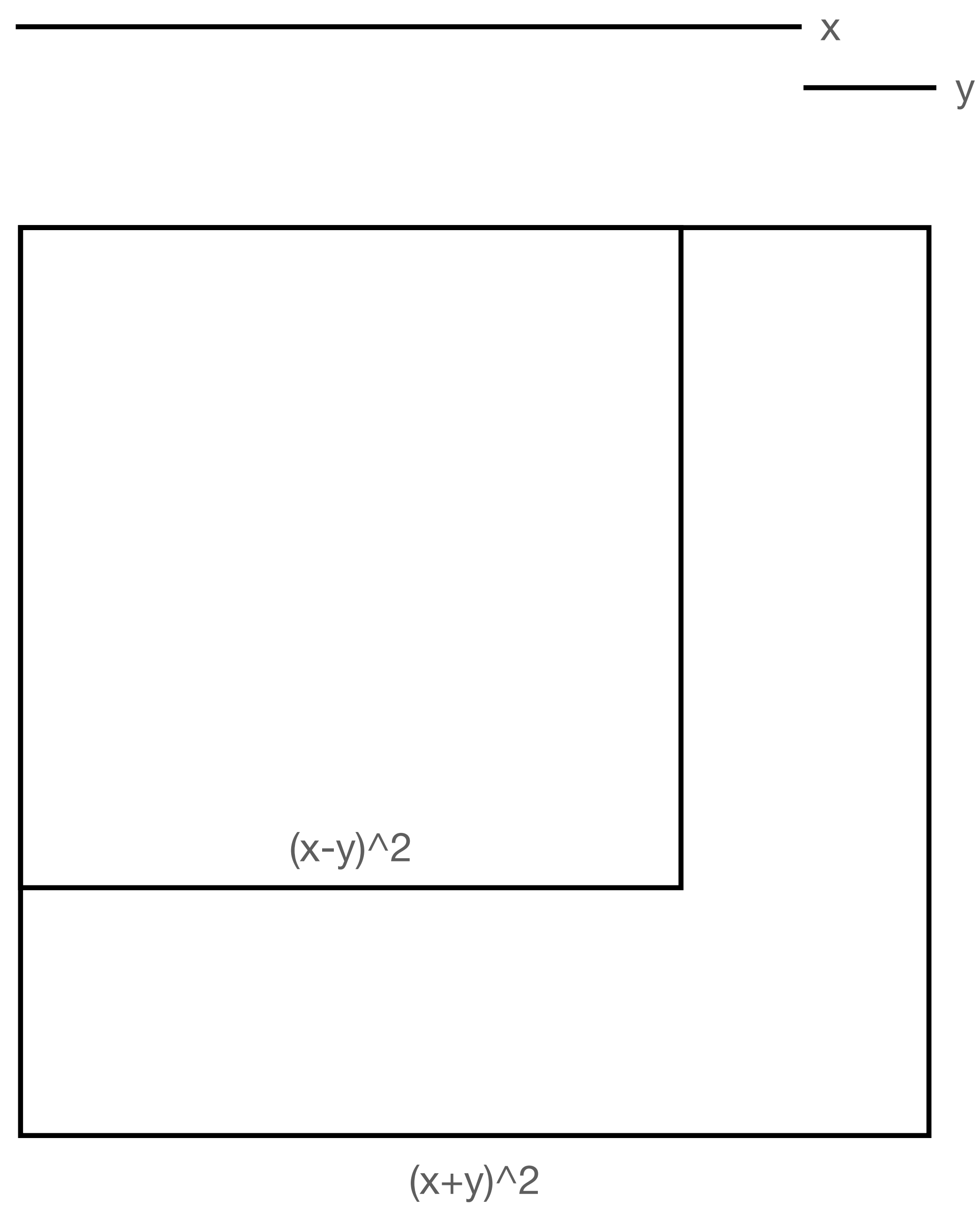

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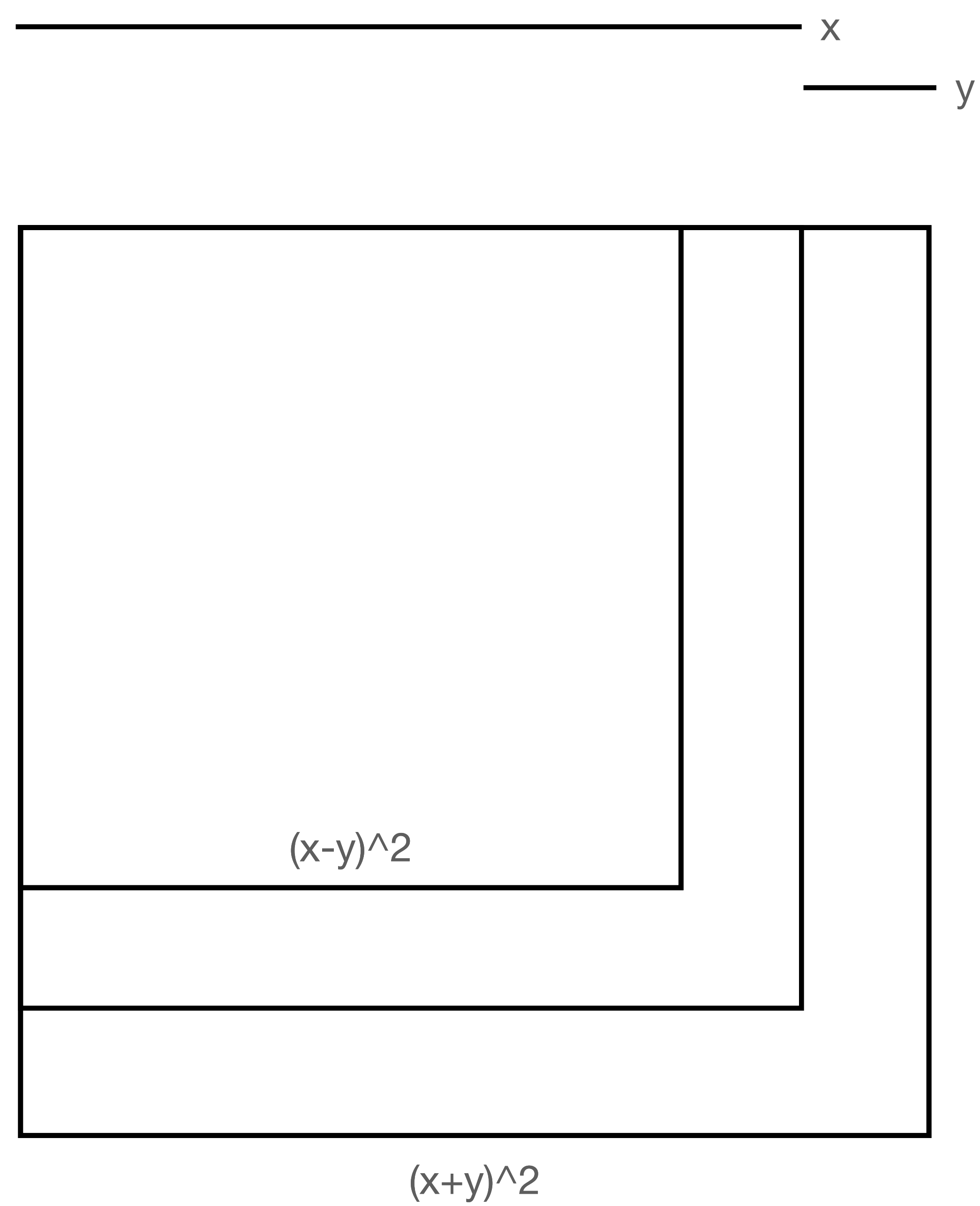

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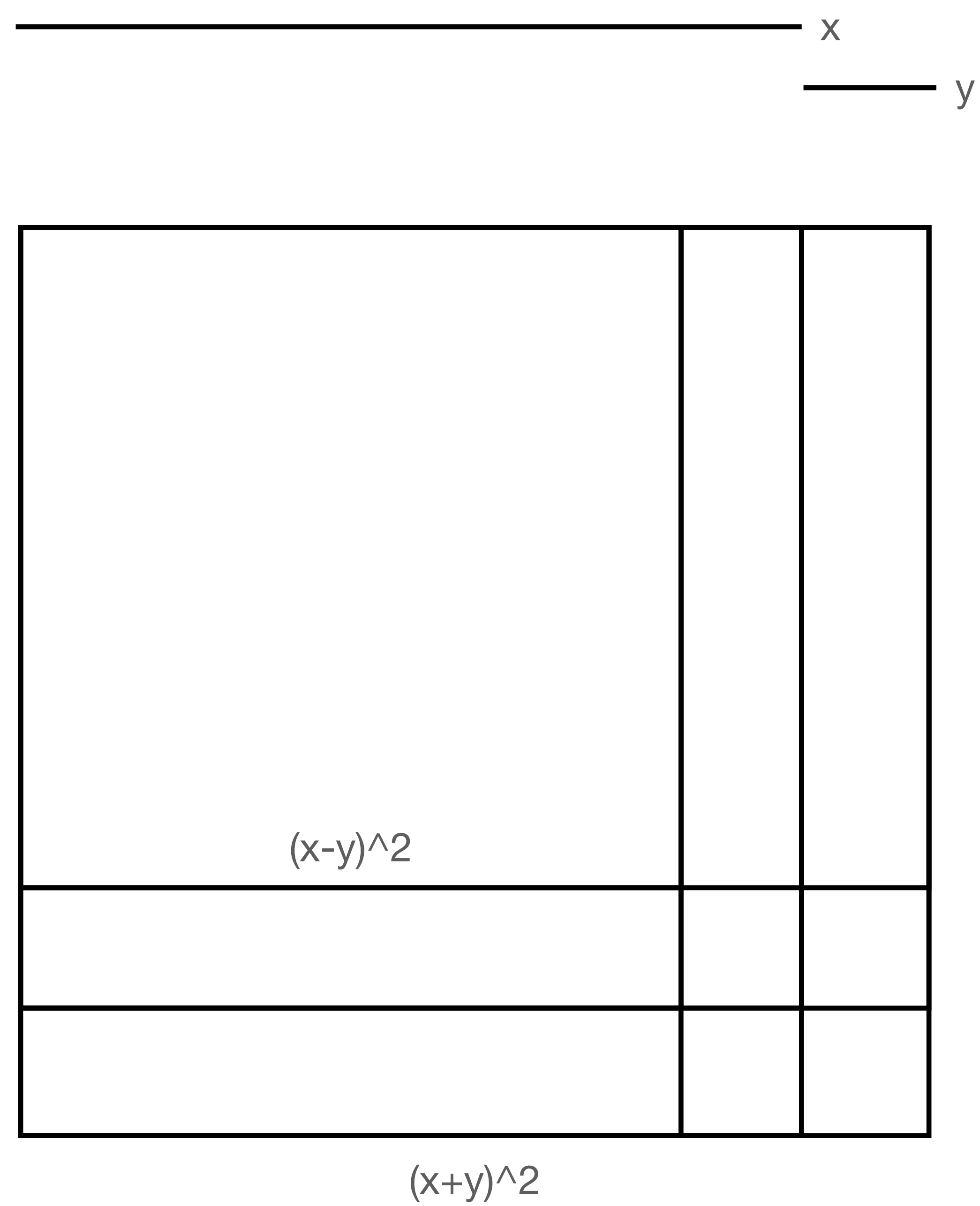

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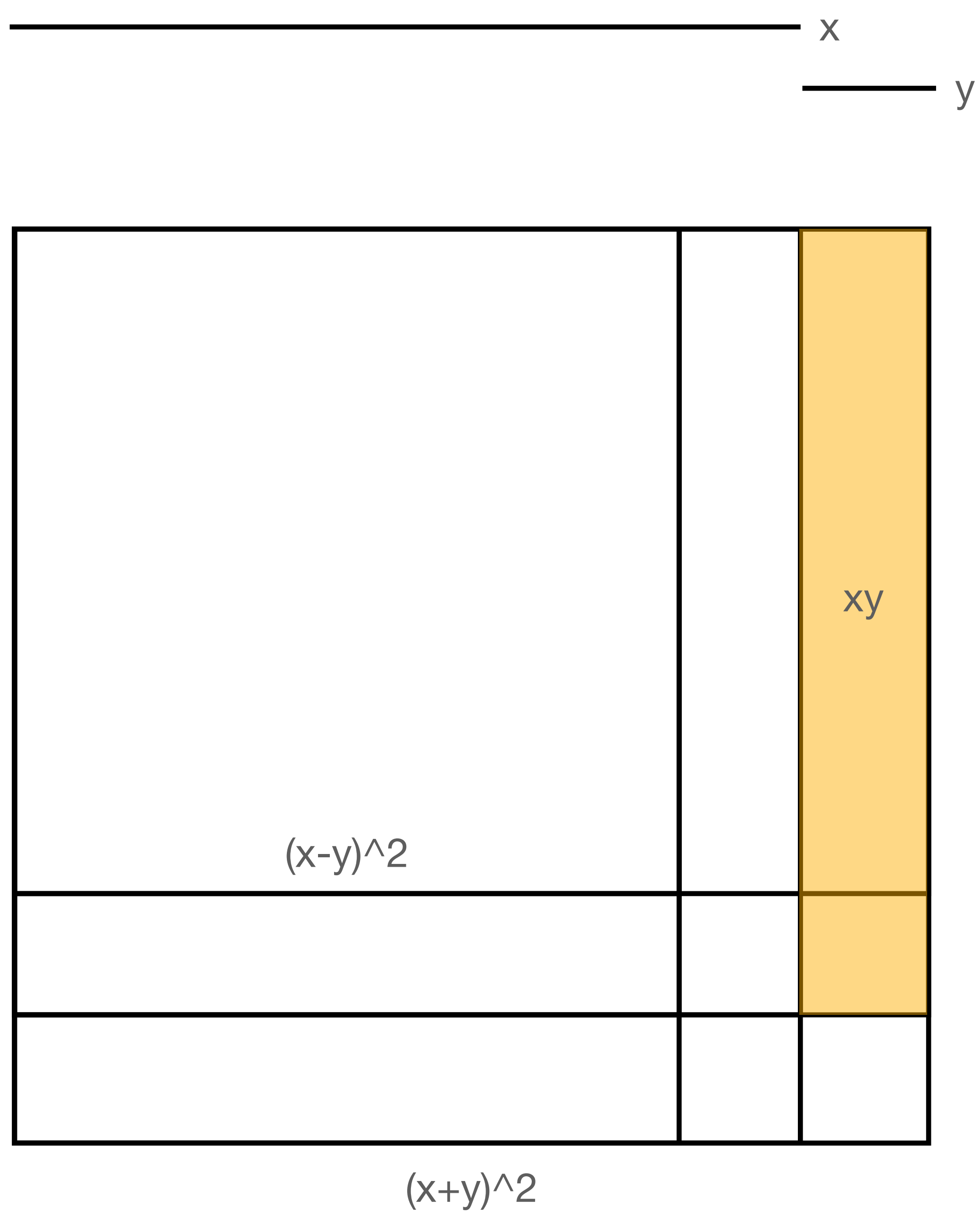

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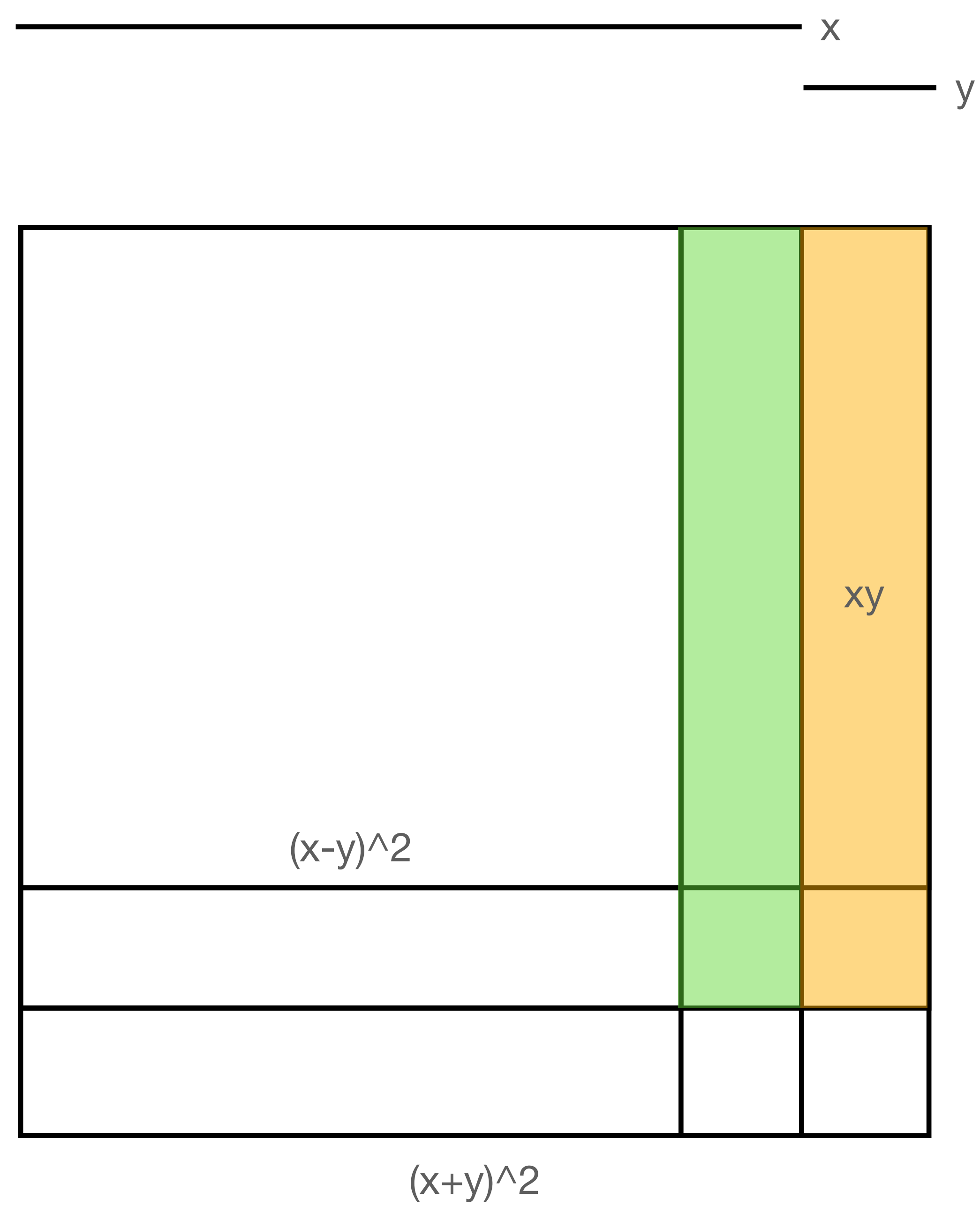

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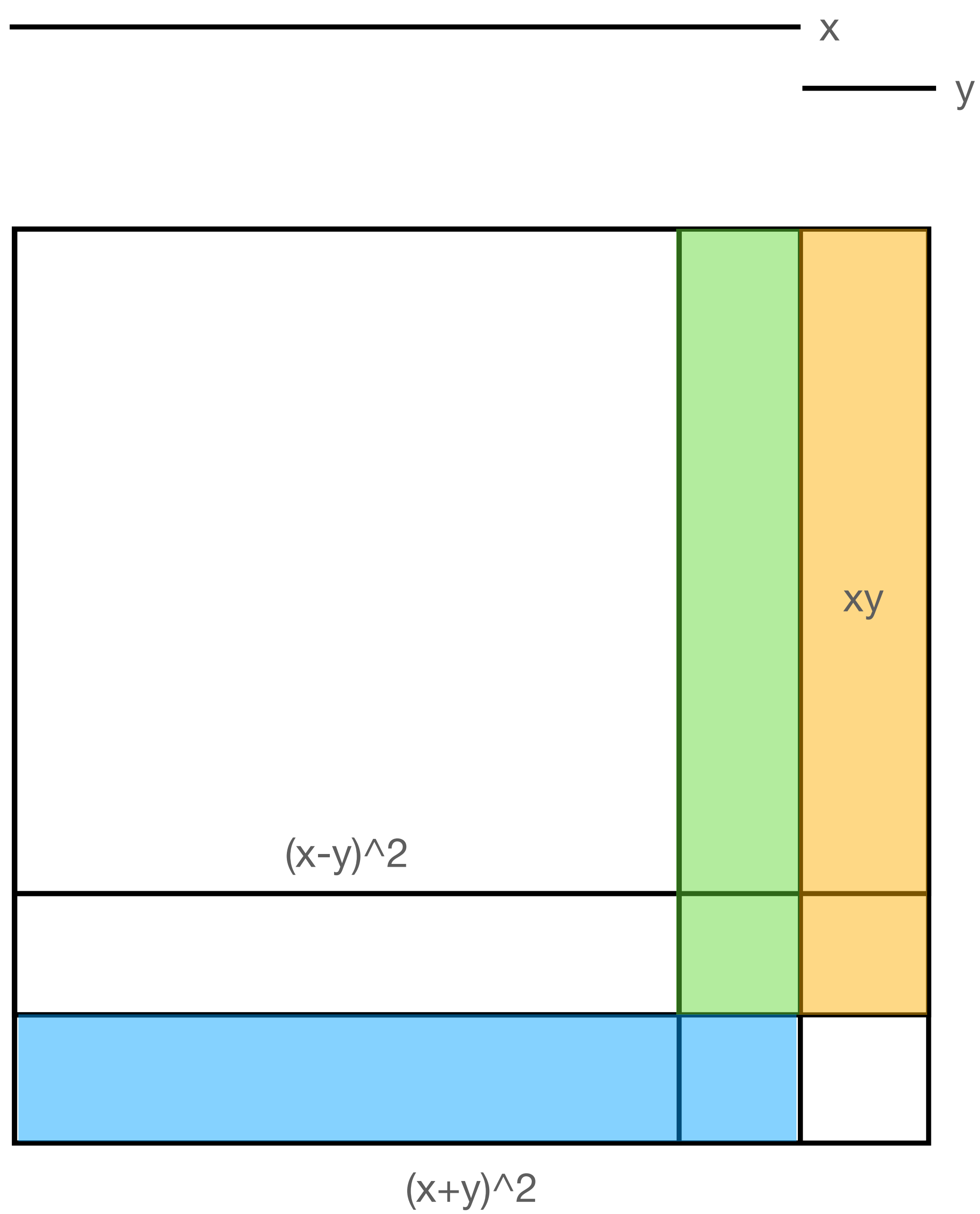

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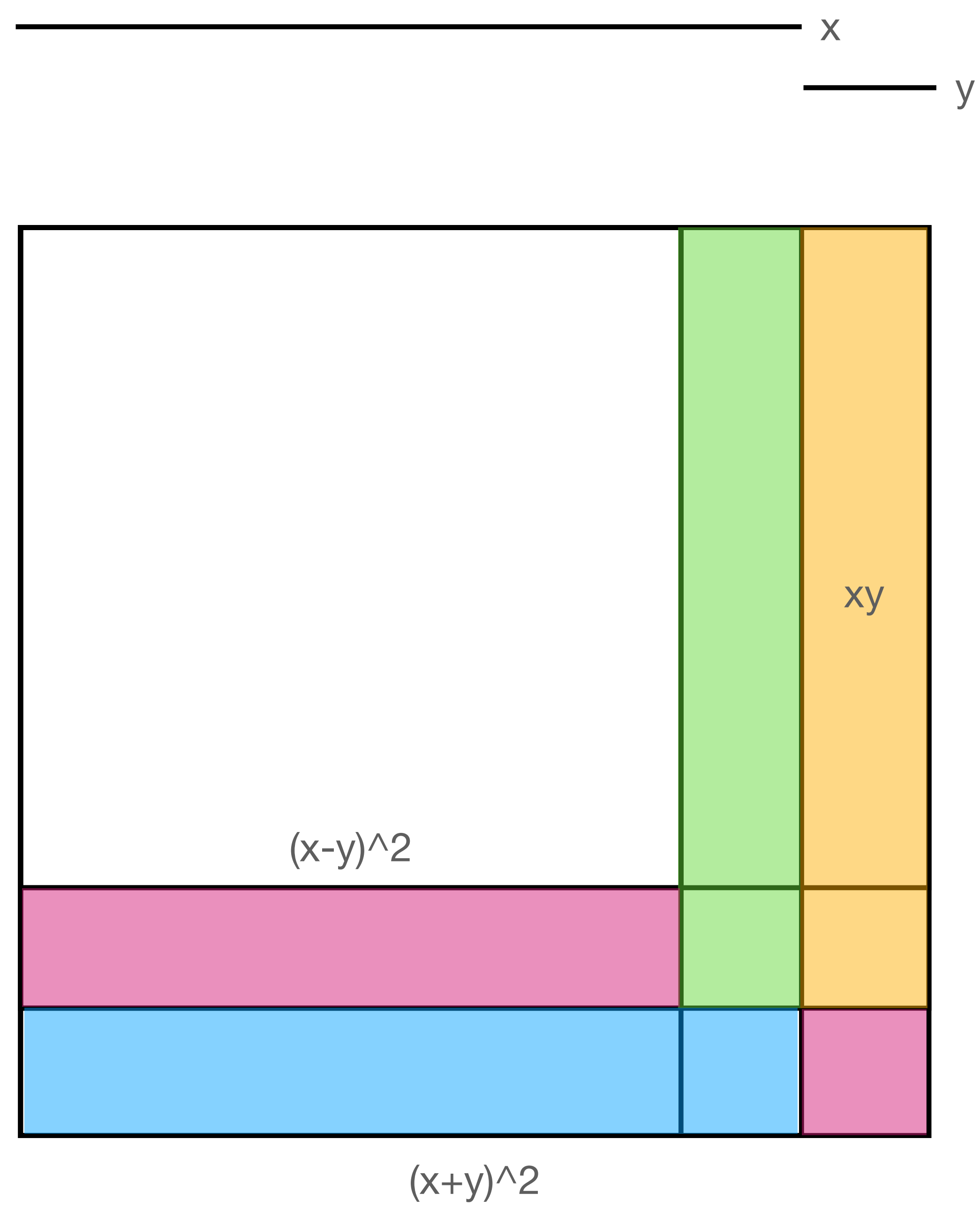


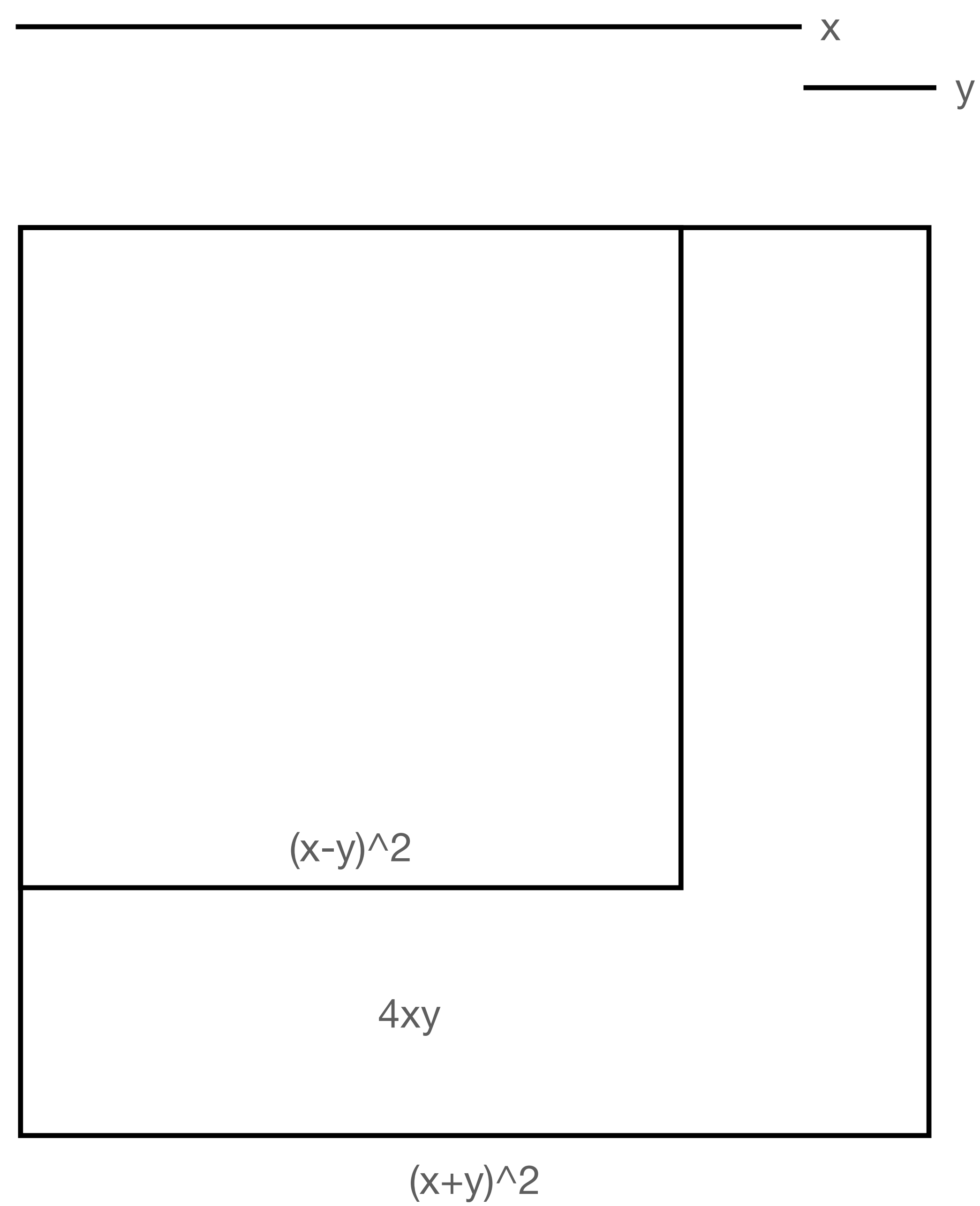
Where are these random solutions coming from?

$$(a-b)^2 + 4ab = (a+b)^2$$

If you realize quadratic defined by

$$a + b = -p$$
$$ab = q$$

The solution is right there



(Euclid II.8)

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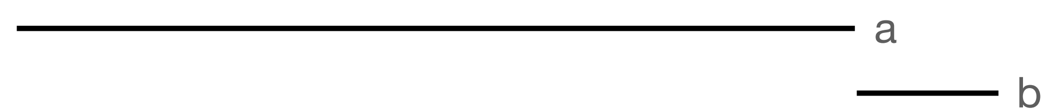

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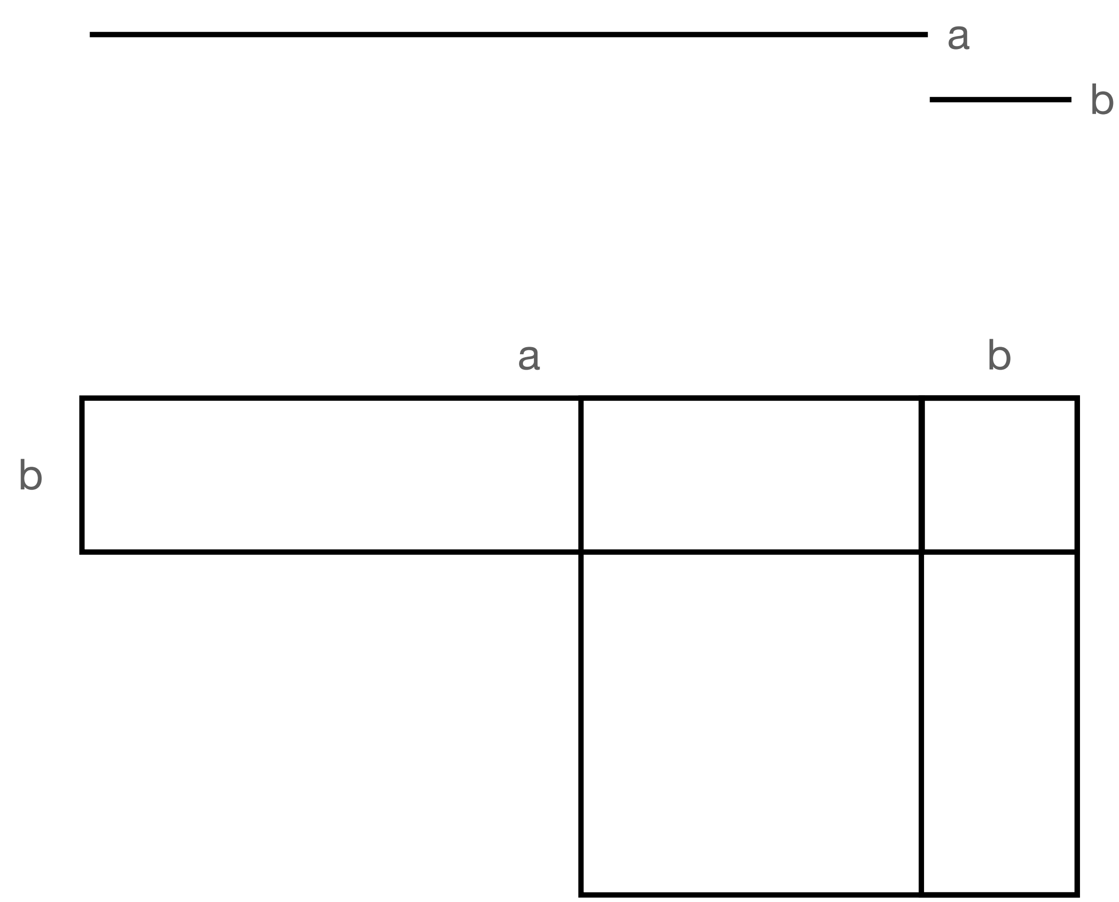

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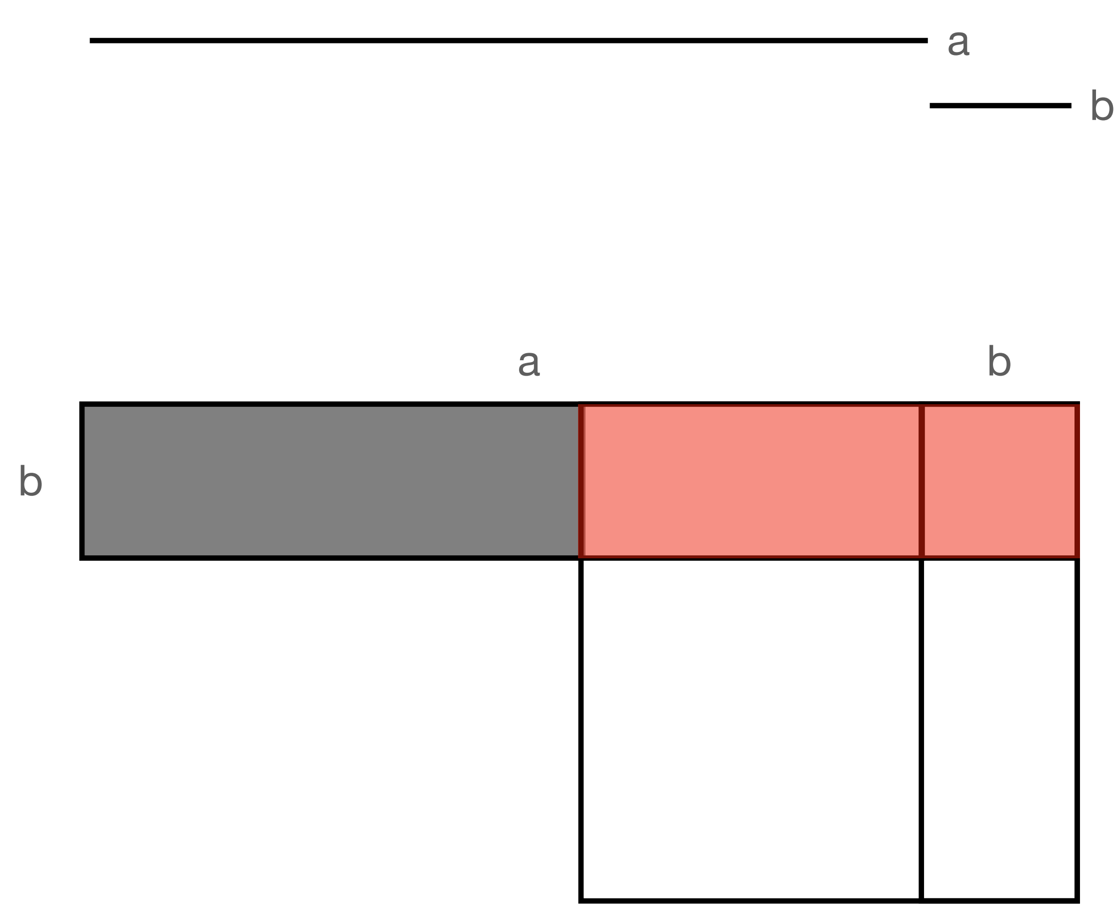

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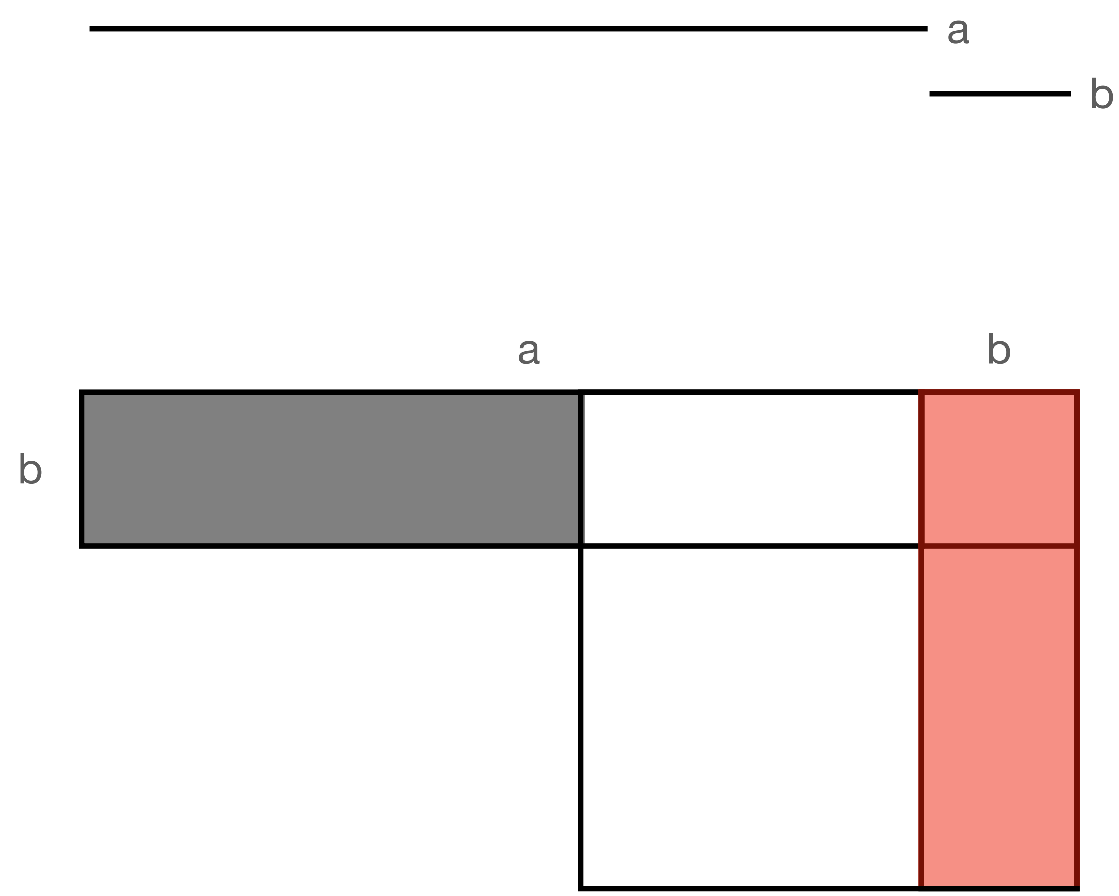

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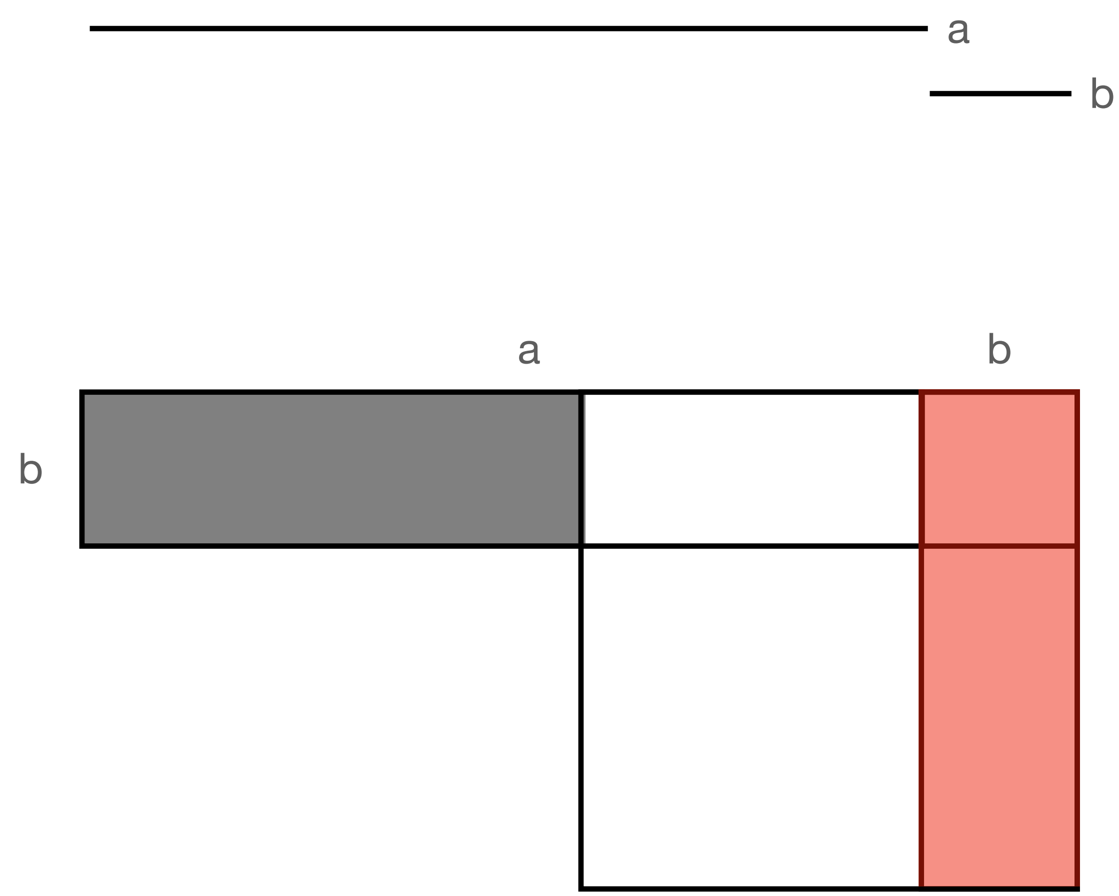

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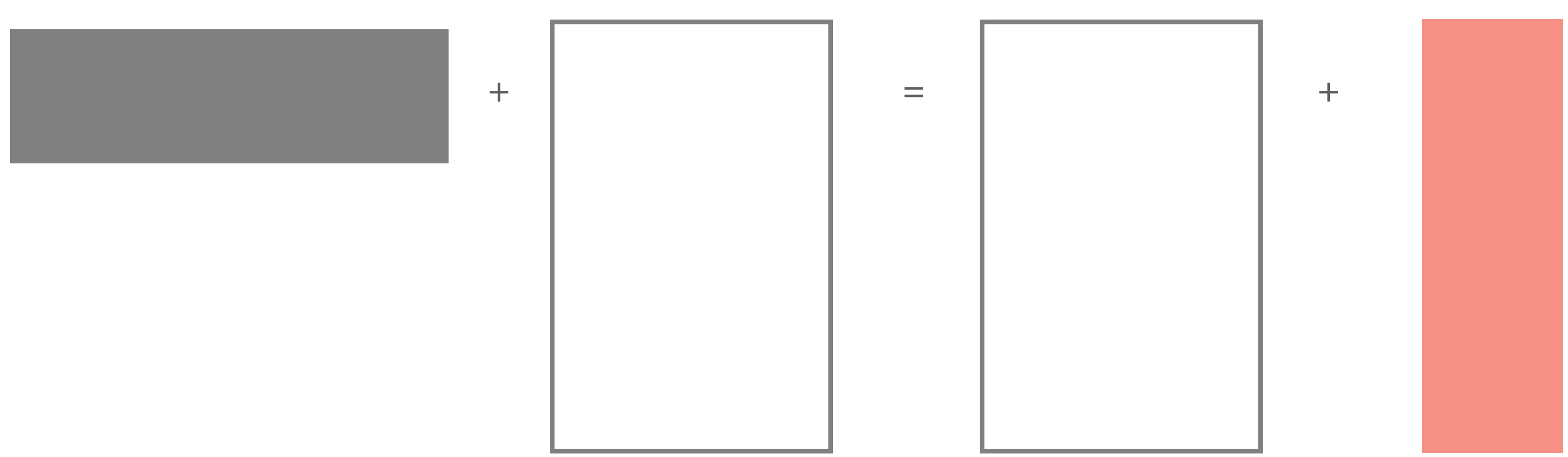

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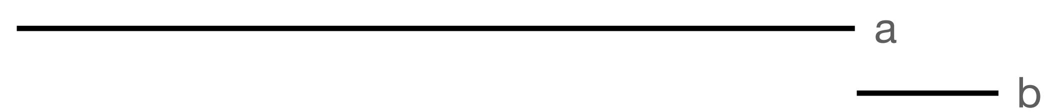

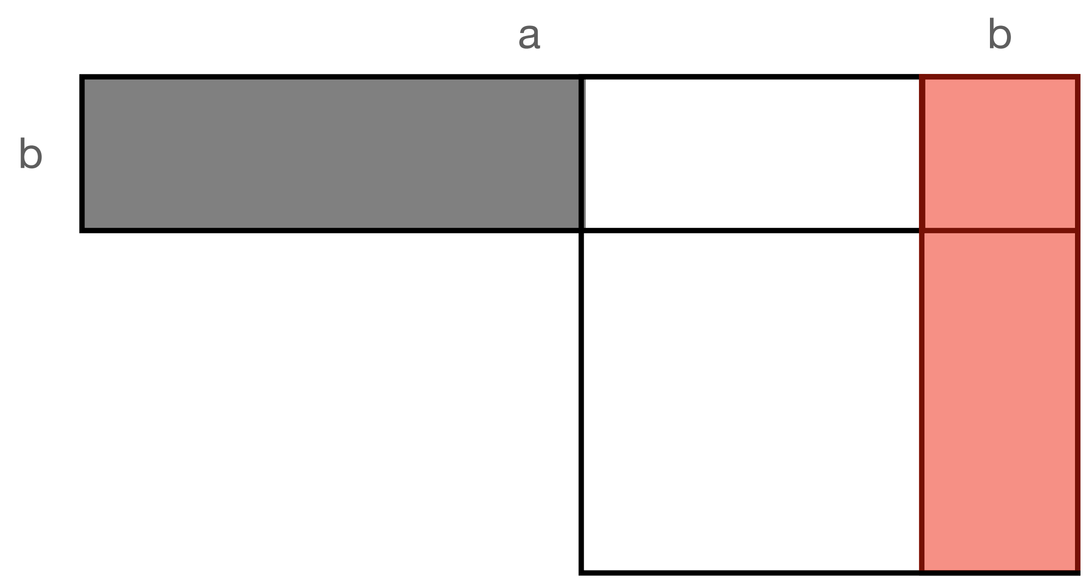

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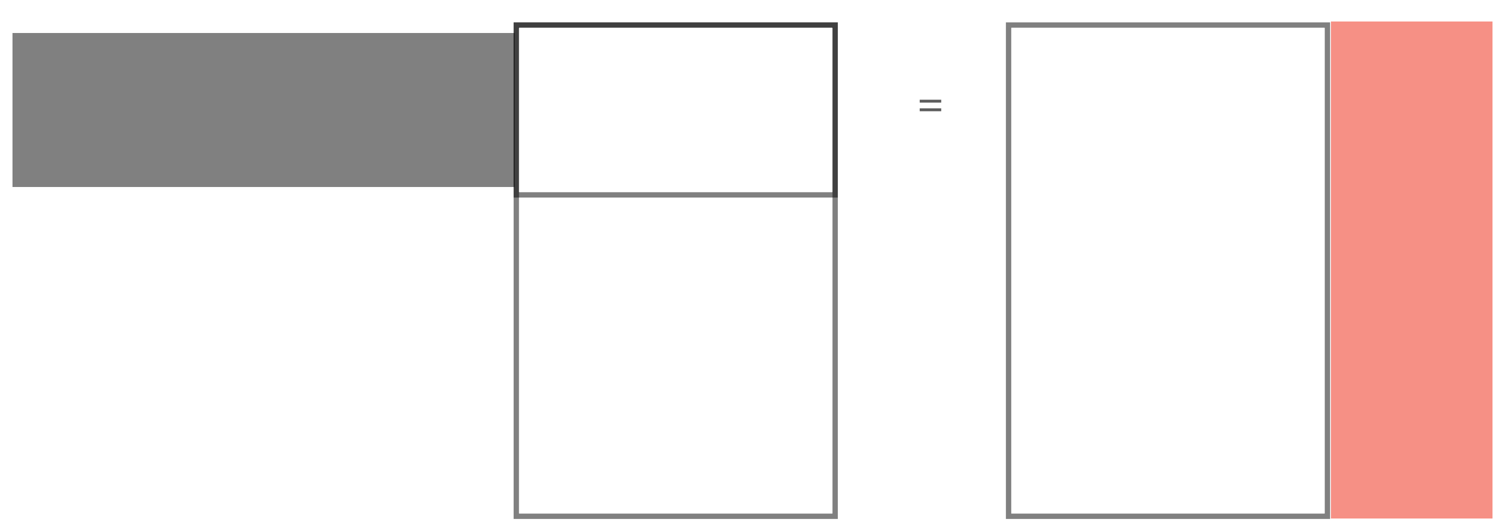

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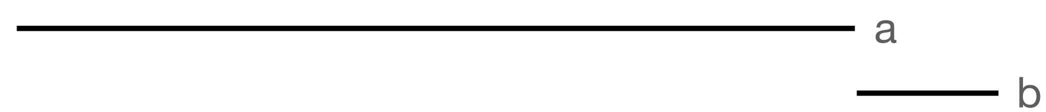

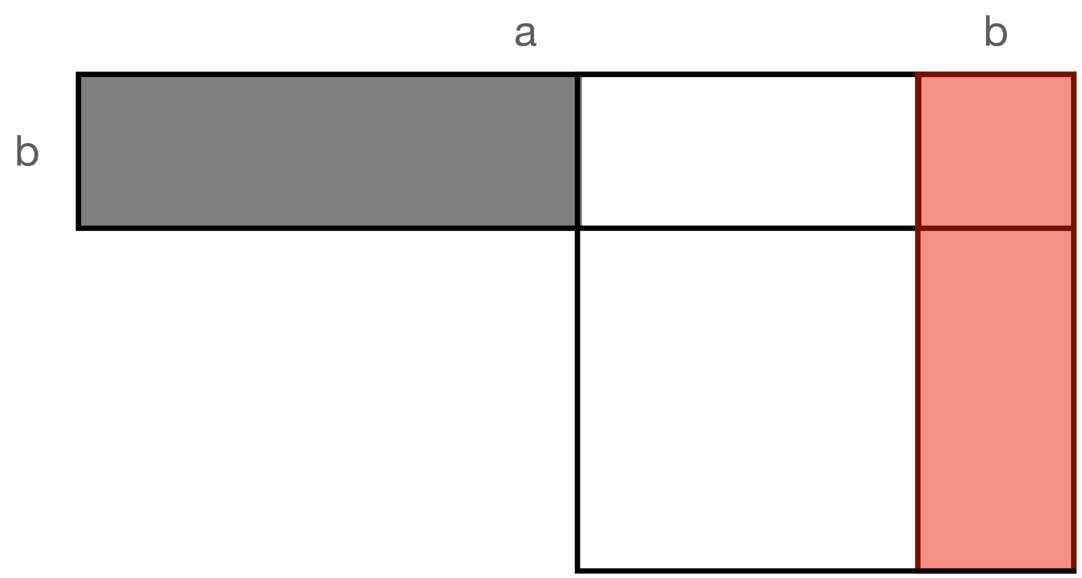

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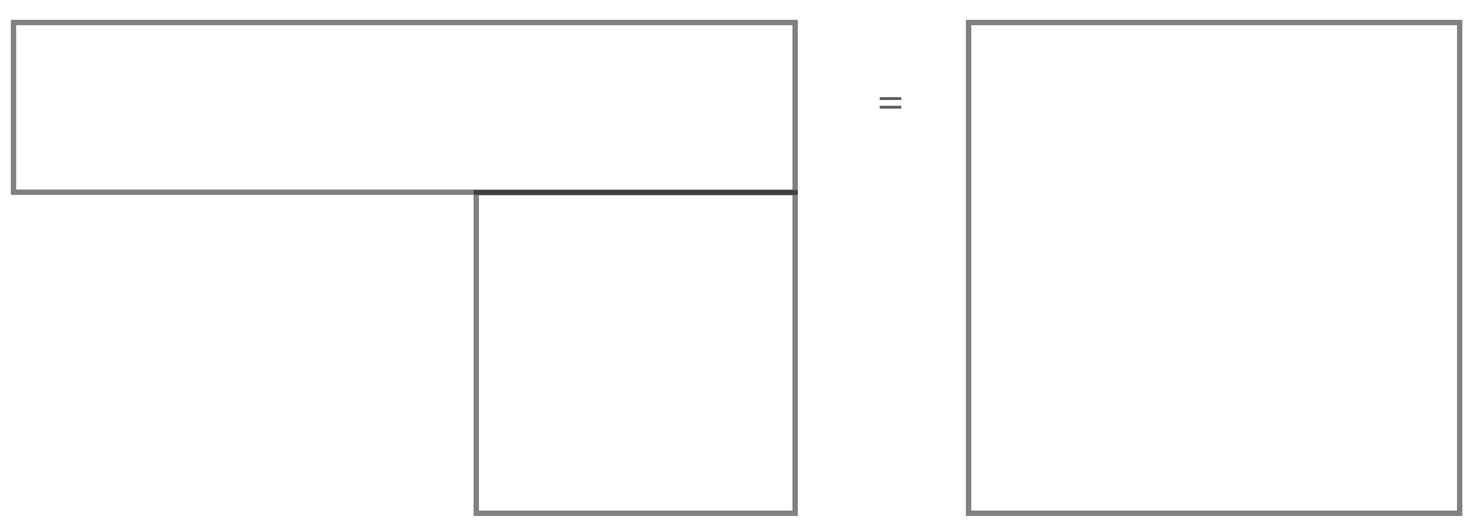

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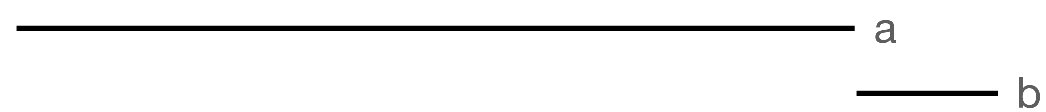
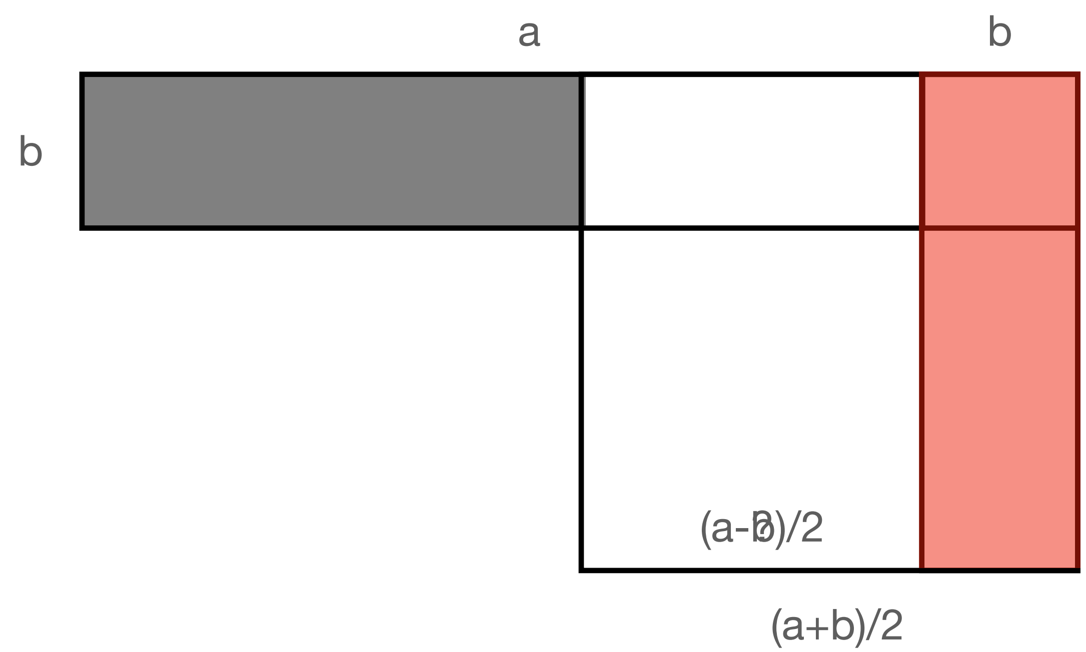




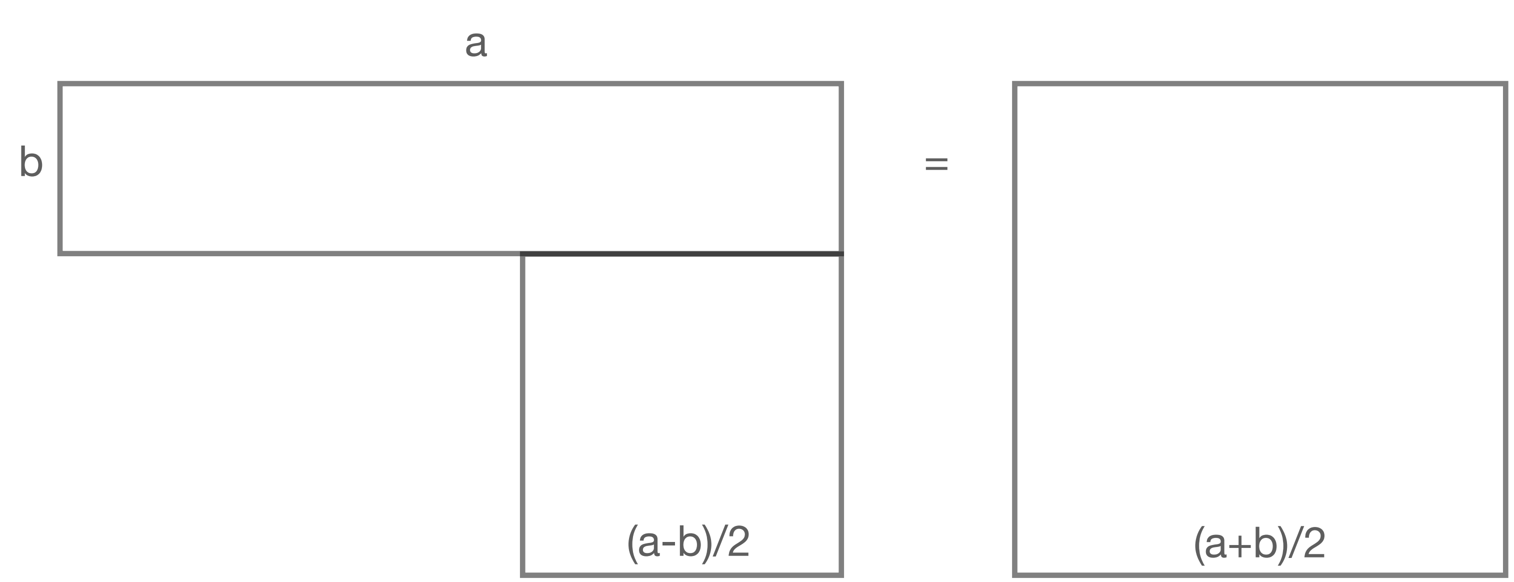
? = (a+b)/2 - b

? = (a+b)/2 - 2b/2

? = (a-2)/2

$$a + b = -p$$

$$ab = q$$



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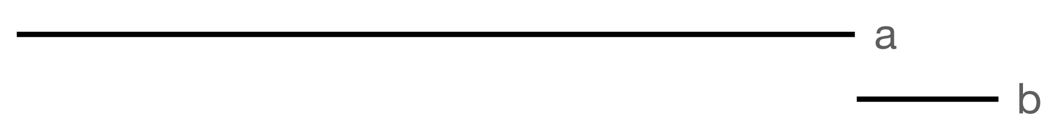
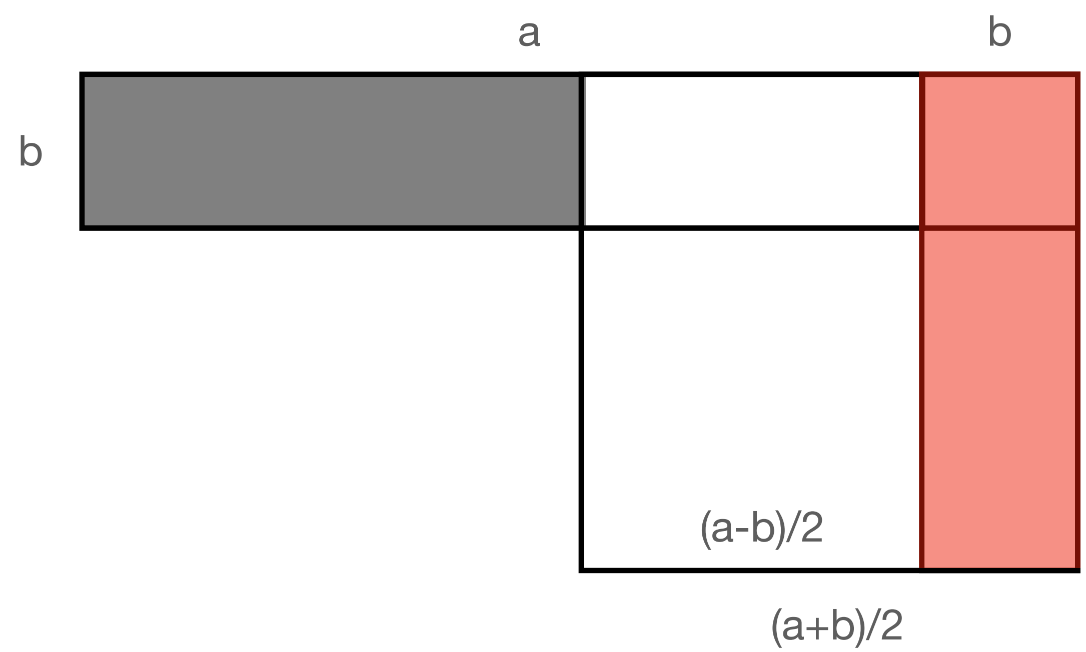




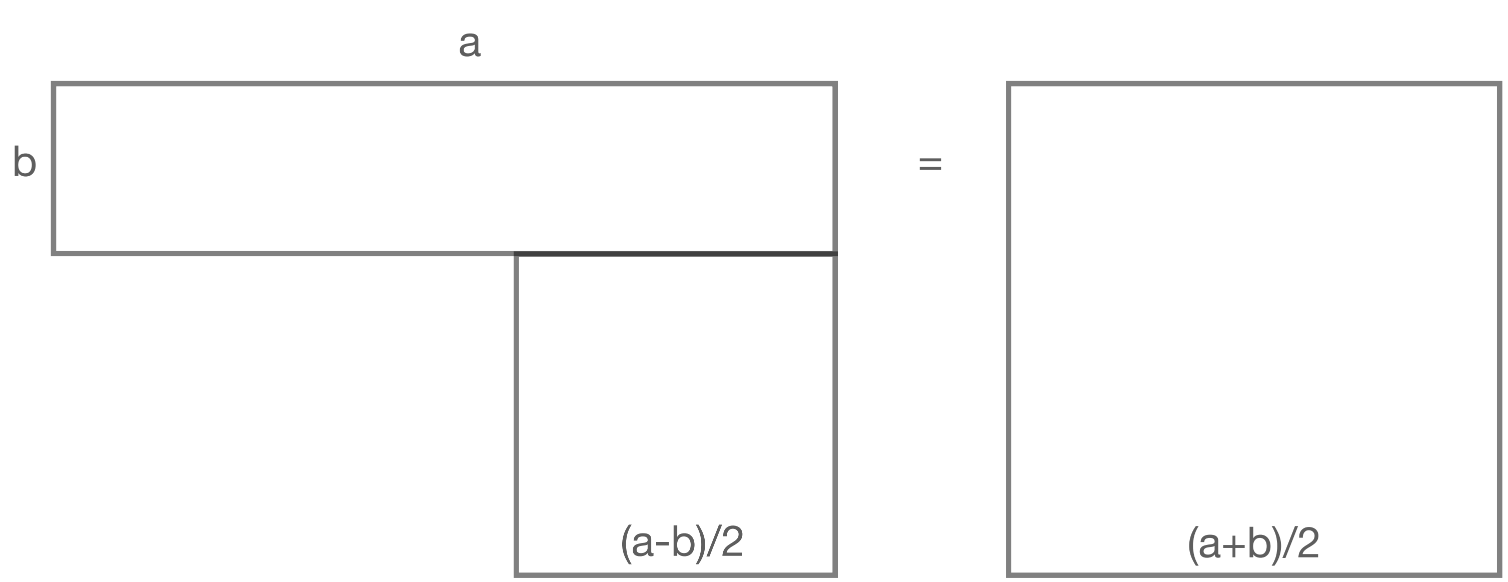
$$a + b = -p$$

$$ab = q$$

$$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$$

$$q + h^2 = g^2$$

$$q = g^2 - h^2$$



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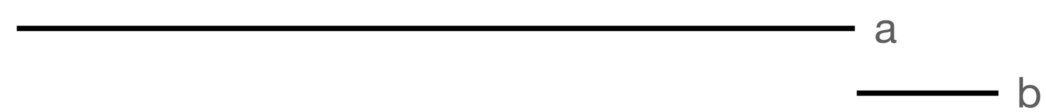

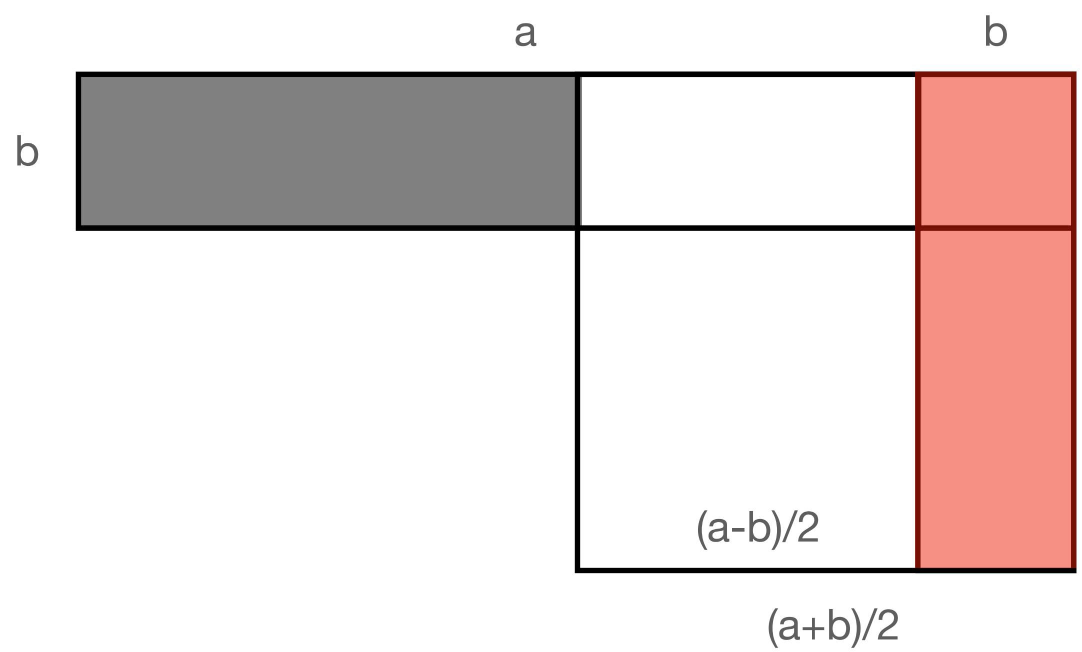

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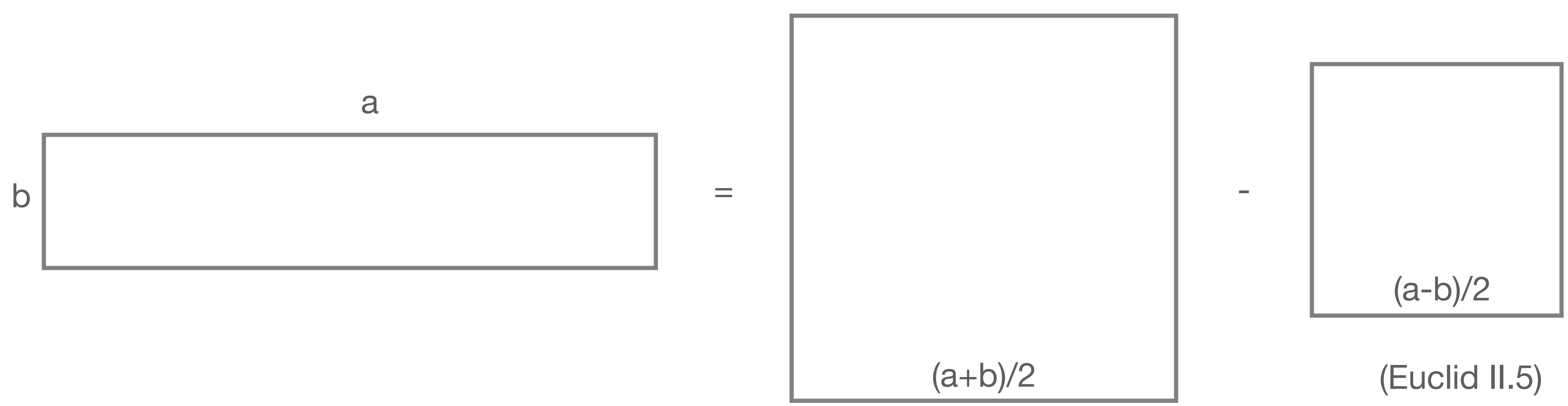


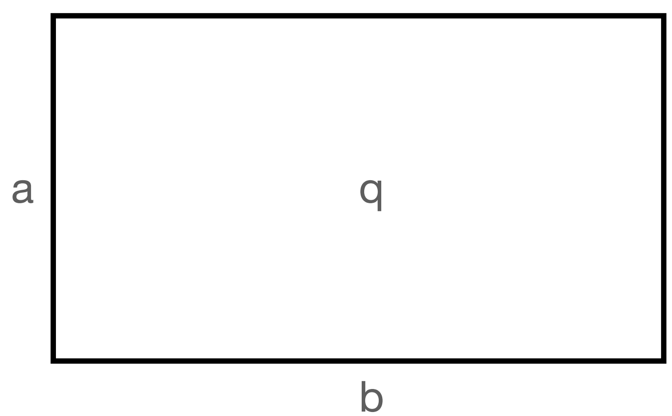
(Euclid II.5)

Where are these random solutions coming from?

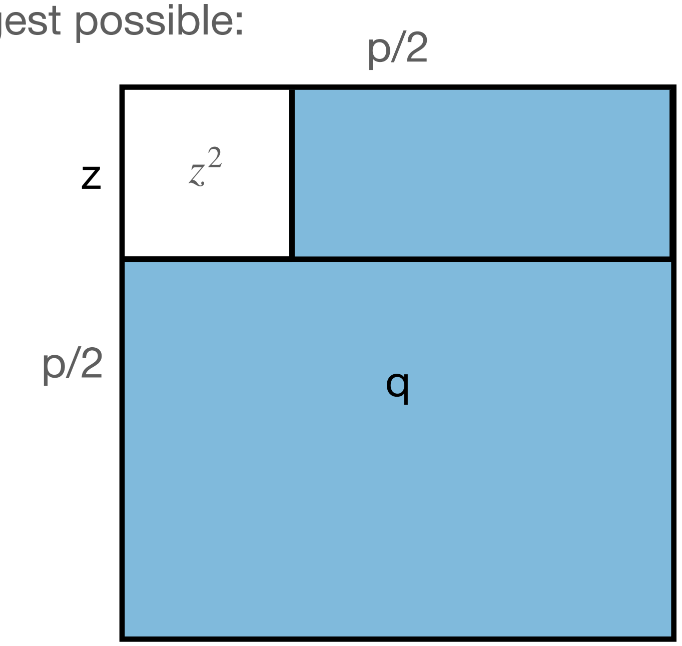
$$x^2 - x(a+b) + ab = 0$$

$$a + b = p$$

$$ab = q$$



Biggest possible:



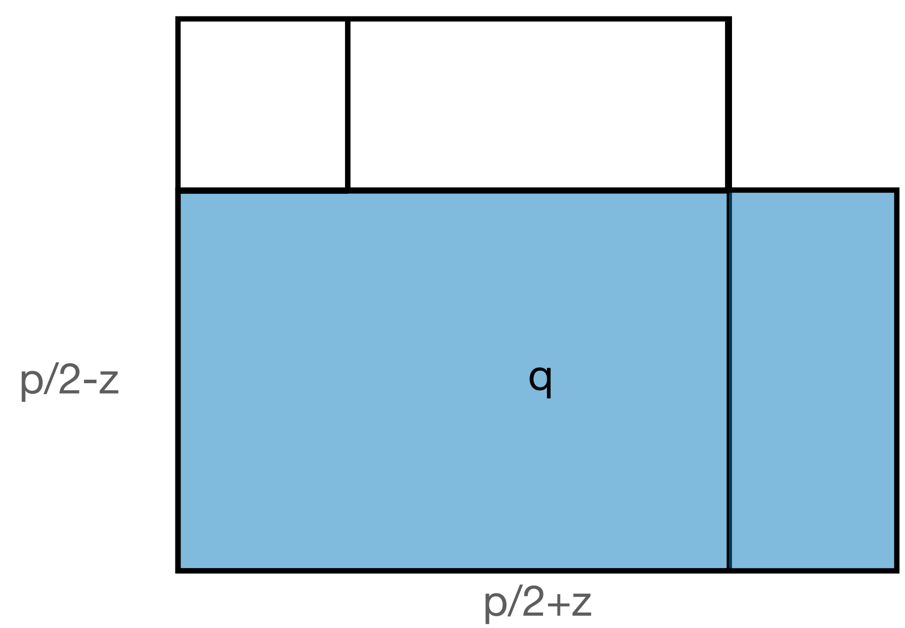
$$a, b = \frac{p}{2}$$

if q too big for this,
this square can't exist
with that p

i.e., geometry requires real solutions

$$\frac{p^2}{4} - q \geq 0$$

Solution:



$$z^2 = \frac{p^2}{4} - q$$

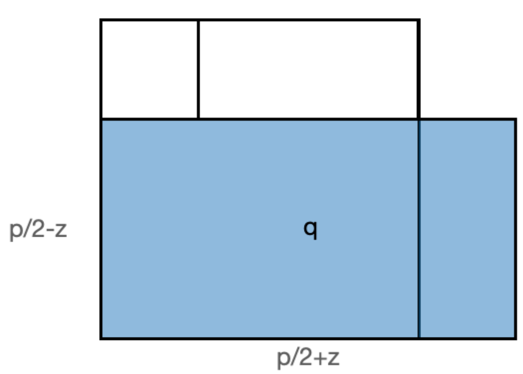
$$a, b = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} \qquad z = \sqrt{\frac{p^2}{4} - q}$$

Again we arrive at the quadratic formula

Many roads lead to the quadratic formula; some illuminate the situation more than others

Brute force

Quadratic formula



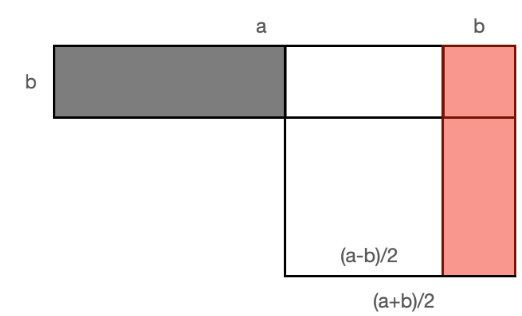
Clever, involved

(abc)
(pq)
q’

Rest on insight that quadratic defined by sum and product

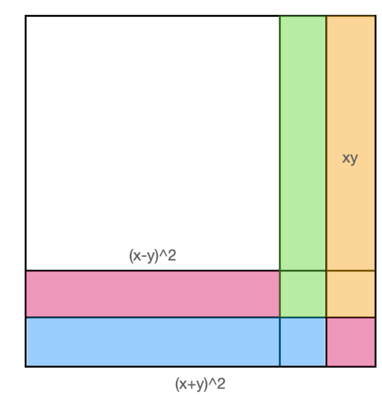
gh: mean, variance

g=(b+a)/2
h=(b-a)/2



sum and diff of squares

(a+b)^2 = (a-b)^2+4ab



The way we receive a formula (e.g., abc, pq) is often not its “native” form (gh)

Note: we got to the depressed q’ form via a change of variable, which was x = y - p/2.

Look familiar? What is the link between the depressed (q’) form and the (gh) form?

Quadratics are fundamentally this:

$$(x-a)(x-b)=0$$

$$x^2-ax-bx+ab=0$$

$$x^2-x(a+b)+ab=0$$

$$x^2+px+q=0$$

$$a+b=-p$$

$$ab=q$$

Depressed quadratic

$$x^2+px+q=0 \qquad x=y+\delta$$

$$(y+\delta)^2+p(y+\delta)+q=0$$

$$y^2+2y\delta+\delta^2+py+p\delta+q=0$$

$$y^2+y(2\delta+p)+(\delta^2+p\delta+q)=0$$

$$\delta=-\frac{p}{2}$$

$$y^2+y(0)+\frac{p^2}{4}-\frac{p^2}{2}+q=0$$

$$y^2+q'=0 \qquad q'=q-\frac{p^2}{4}$$

This yielded simplest solution

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$$y^2+y(2\delta+p)+(\delta^2+p\delta+q)=0 \qquad \delta=-\frac{p}{2}$$

$$y^2+y(0)+\frac{p^2}{4}-\frac{p^2}{2}+q=0 \qquad y^2+q'=0$$

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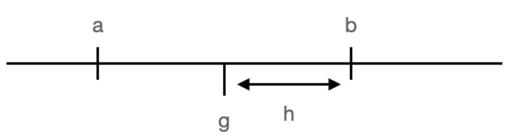
This yielded simplest solution

Reframe in (gh)

$$a=g-h$$

$$b=g+h$$



$$g=\frac{b+a}{2}$$

$$h=\frac{b-a}{2}$$

$$g=-\frac{p}{2} \qquad g^2-h^2=q$$

The way we receive a formula (e.g., abc, pq) is often not its “native” form (gh)

Note: we got to the depressed q’ form via a change of variable, which was x = y - p/2.

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$$y^2+y(2\delta+p)+(\delta^2+p\delta+q)=0$$

$$y^2+y(0)+\frac{p^2}{4}-\frac{p^2}{2}+q=0$$

This yielded simplest solution

$$x=y+\delta$$

$$x=y+g$$

$$\delta=-\frac{p}{2}$$

$$y^2+q'=0$$

$$q'=q-\frac{p^2}{4}$$

$$g=-\frac{p}{2}$$

$$g=\delta$$

Reframe in (gh)

$$a=g-h$$

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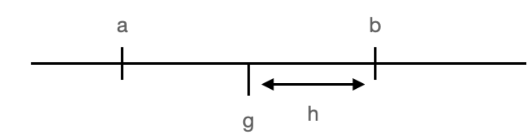

$$g=\frac{b+a}{2}$$

$$h=\frac{b-a}{2}$$

$$g^2-h^2=q$$

The way we receive a formula (e.g., abc, pq) is often not its “native” form (gh)

Note: we got to the depressed q’ form via a change of variable, which was x = y - p/2.

Look familiar? What is the link between the depressed (q’) form and the (gh) form?

The (q’) form <u>IS</u> the (gh) form - it is the “native” form and solution for the situation

Quadratics are fundamentally this:

$$(x-a)(x-b)=0$$

$$x^2-ax-bx+ab=0$$

$$x^2-x(a+b)+ab=0$$

$$x^2+px+q=0$$

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Depressed quadratic

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$$y^2+y(0)+\frac{p^2}{4}-\frac{p^2}{2}+q=0$$

This yielded simplest solution

$$x=y+\delta$$

$$x=y+g$$

$$\delta=-\frac{p}{2}$$

$$y^2+q'=0$$

$$q'=q-\frac{p^2}{4}$$

$$g=-\frac{p}{2}$$

$$g=\delta$$

$$y^2+(q-g^2)=0$$

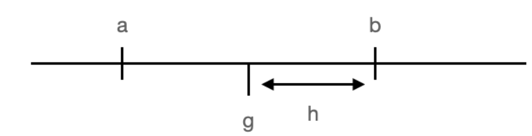
$$q'=q-\left(\frac{-p}{2}\right)^2$$

$$q'=q-g^2$$

Reframe in (gh)

$$a=g-h$$

$$b=g+h$$



$$g=\frac{b+a}{2}$$

$$h=\frac{b-a}{2}$$

$$g^2-h^2=q$$

$$h=y$$

$$g^2-y^2=q$$

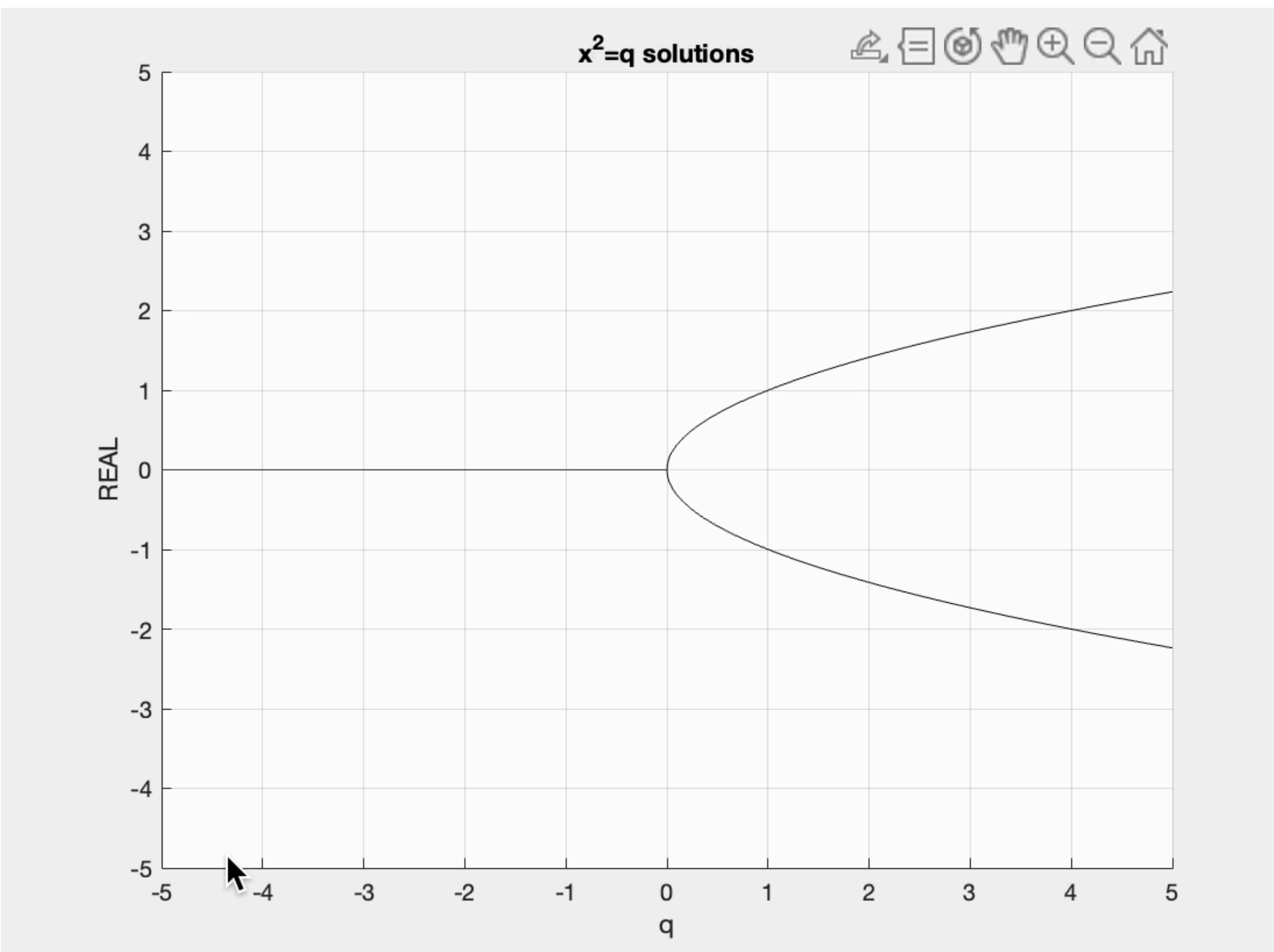
$$q=g^2-y^2$$

The way we receive a formula (e.g., abc, pq) is often not its “native” form (gh)

Note: we got to the depressed q’ form via a change of variable, which was x = y - p/2.

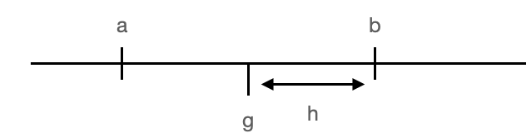
Look familiar? What is the link between the depressed (q’) form and the (gh) form?

The (q’) form IS the (gh) form - it is the “native” form and solution for the situation



Reframe in (gh)

$$a = g - h$$
$$b = g + h$$



$$g = \frac{b+a}{2}$$
$$h = \frac{b-a}{2}$$

$$x = y + \delta$$
$$x = y + g$$

$$\delta = -\frac{p}{2} \qquad g = -\frac{p}{2} \qquad g^2 - h^2 = q$$

$$g = \delta \qquad h = y$$

$$g^2 - y^2 = q$$

$$y^2 + q' = 0 \qquad y^2 + (q - g^2) = 0 \qquad q = g^2 - y^2$$

$$q' = q - \frac{p^2}{4} \qquad q' = q - \left(\frac{-p}{2}\right)^2$$

$$q' = q - g^2$$

Why dwell on quadratics like this?

It’s a great sandbox, and the entry point to some sophisticated math

Illustrates that the more ways you can approach a problem, the better your understanding (and elegance of solutions)

Techniques

Change of variable: $x=y+d$

Redefine variables symmetrically: gh as [mean, difference]

Invert squared expression: $(a+b)^2=4xy+(a-b)^2$

Often, there is a geometric analogue of an algebraic situation

Questions:

Q1: “Find two numbers whose sum is M, and whose product is N”

$$a + b = M \qquad\qquad ab = N$$

Starting from this classic situation, derive the quadratic this represents, and how to solve it. You DO NOT need another variable.

Questions:

Q2: “Find two numbers whose sum is M, and whose product is N”

If you imagine all numbers x,y as possible solutions, can you sketch the situation above in 3D?

Do all (M,N) produce a solution in your sketch? How does this relate to the (pq) surface?

Q3: Draw the 3D geometric object: $(a+b)^3$

Expand algebraically and relate that expansion to the 3D object.

Q4: Suppose I said that (pq) were complex numbers. (totally legit)

Can you imagine the solution space still?

Does it help to go to (q')?

Does it help to drop powers?

i.e., could we "see" $y=bx$ if b were complex?

Q5: Our perimeter problem solved a situation where M and N below are positive

$$a + b = M \qquad\qquad ab = N$$

Geometric solutions only work for positive values

Suppose p and q are positive in the following forms, can you find corresponding geometric models?

$$x^2 + q = px \qquad x^2 = px + q \qquad x^2 = q \qquad x^2 = px \qquad x^2 + px = q$$