Mental exercise: geometry and complex numbers

Solve these equations:

Solve these equations:

$x^2 = 1$

Solve these equations:

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$x^2 = -1$

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$x^2 = 1$ $x^2 = -1$ $x^4 = 1$

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Solve these equations:

$x^2 = 1$

$x^2 = -1$

$x^4 = 1$

$x^3 = 1$

$x^2 - 1 = 0$

Solve these equations:

$$x^2 = 1$$

$$x^2 - 1 = 0$$

$$(x + 1)(x - 1) = 0$$

$$x^2 = -1$$

$$x^4 = 1$$

$$x^3 = 1$$

Solve these equations:

$x^2 = 1$

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1, 1$

$x^2 = -1$

$x^4 = 1$

$x^3 = 1$

Solve these equations:

$x^2 = 1$

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1, 1$

$x^2 = -1$

$x = \pm i$

$x^4 = 1$

$x^3 = 1$

Solve these equations:

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$$x^2 - 1 = 0$$

$$(x + 1)(x - 1) = 0$$

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$$x = \pm i$$
$$x^2 + 1 = 0$$
$$(x - i)(x + i) = 0$$

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$$x = \pm 1, \pm i$$

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$$x^3 - 1 = 0$$
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$$x = 1$$

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$$x = \pm 1, \pm i$$

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$$x^3 - 1 = 0$$
$$(x-1)(x^2+x+1) = 0 \qquad x = 1$$
$$(x^2+x+1) = 0$$

Solve these equations:

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$x = -1, 1$

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$(x-1)(x+1)(x^2+1) = 0$

$(x-1)(x+1)(x-i)(x+i) = 0$

$x = \pm 1, \pm i$

$x^3 = 1$

$x^3 - 1 = 0$

$(x-1)(x^2+x+1) = 0$ $\quad x = 1$

$(x^2+x+1) = 0$

Use quadratic formula

Solve these equations:

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$$x^2 - 1 = 0$$
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Use quadratic formula

$$x = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2}$$

Solve these equations:

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$$x = -1, 1$$

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$$x^3 - 1 = 0$$
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Use quadratic formula

$$x = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2}$$
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Solve these equations:

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$$x = \pm 1, \pm i$$

$$x^3 = 1$$
$$x^3 - 1 = 0$$
$$(x-1)(x^2+x+1) = 0 \qquad x = 1$$
$$(x^2+x+1) = 0$$

Use quadratic formula

$$x = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2}$$
$$x = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$
$$x = \frac{-1 \pm \sqrt{3}i}{2}, 1$$

Solve these equations:

$$x^2 = 1$$
$$x^2 - 1 = 0$$
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$$x = -1, 1$$

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$$x = \pm 1, \pm i$$

$$x^3 = 1$$
$$x^3 - 1 = 0$$
$$(x-1)(x^2+x+1) = 0 \qquad x = 1$$
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Use quadratic formula

$$x = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2}$$
$$x = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$
$$x = \frac{-1 \pm \sqrt{3}i}{2}, 1$$

These examples illustrate:

1) how annoying cubics can be to solve
2) how useful conjugates and quadratic solutions are
3) that polynomials of order N have N roots, which may be real or complex

With only counting numbers

With only counting numbers $x_i \in \{0,1,2,3,4...\}$

Which are always solvable?

With only counting numbers $x_i \in \{0,1,2,3,4...\}$

Which are always solvable? $x_1 + x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$

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Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

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d is a number built from N

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

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Invent negative integers

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

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Invent negative integers $x_i \in \mathbb{Z}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = n x_2 + r$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = n x_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = n x_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = n x_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \dfrac{x_1}{x_2}$

e is a fraction built from Z

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = n x_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

e is a fraction built from Z

Invent fractions

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

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Invent negative integers $x_i \in \mathbb{Z}$

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With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = n x_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

e is a fraction built from Z

Invent fractions $x_i \in \mathbb{Q}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$ $x^2 = c$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = nx_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

e is a fraction built from Z

Invent fractions $x_i \in \mathbb{Q}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$ $x^2 = c$ $x^2 = -c$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = nx_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

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Invent fractions $x_i \in \mathbb{Q}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$ $x^2 = c$ $x^2 = -c$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

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Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = nx_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

e is a fraction built from Z

Invent fractions $x_i \in \mathbb{Q}$

$c \neq \frac{p^2}{q^2}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$ $x^2 = c$ $x^2 = -c$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = nx_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

e is a fraction built from Z

Invent fractions $x_i \in \mathbb{Q}$

$c \neq \frac{p^2}{q^2}$ $x_1 = \sqrt{c} \notin \mathbb{Q}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$ $x^2 = c$ $x^2 = -c$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

d is a number built from N

Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = nx_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

e is a fraction built from Z

Invent fractions $x_i \in \mathbb{Q}$

$c \neq \frac{p^2}{q^2}$ $x_1 = \sqrt{c} \notin \mathbb{Q}$ $f = \sqrt{c}$

Which are always solvable? $x_1 + x_2$ $x_1 - x_2$ $x_1 \times x_2$ $x_1 \div x_2$ $x^2 = c$ $x^2 = -c$

With only counting numbers $x_i \in \{0,1,2,3,4...\}$ $x_i \in \mathbb{N}$

$x_2 > x_1$ $x_1 - x_2 \notin \mathbb{N}$ $x_1 = d + x_2$

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Invent negative integers $x_i \in \mathbb{Z}$

$x_1 = nx_2 + r$ $x_1 \div x_2 \notin \mathbb{Z}$ $e = \frac{x_1}{x_2}$

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Invent fractions $x_i \in \mathbb{Q}$

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f is a limit that approaches a fraction in Q

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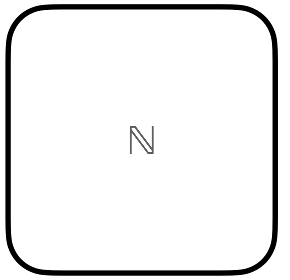
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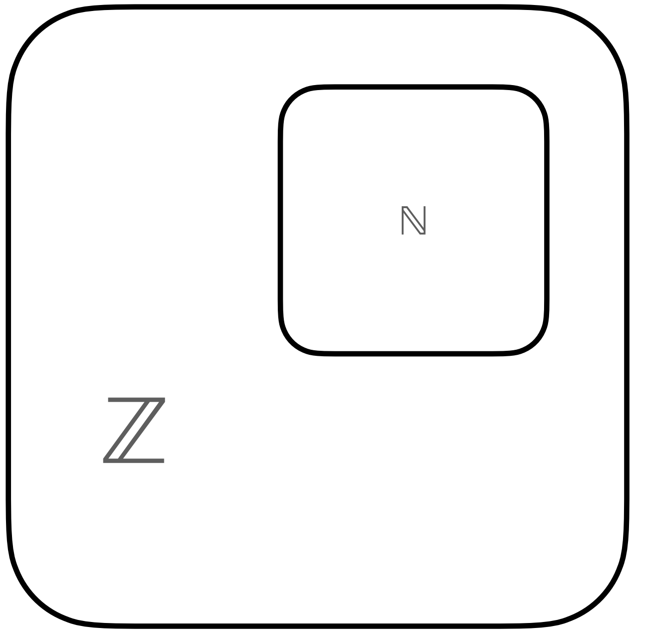
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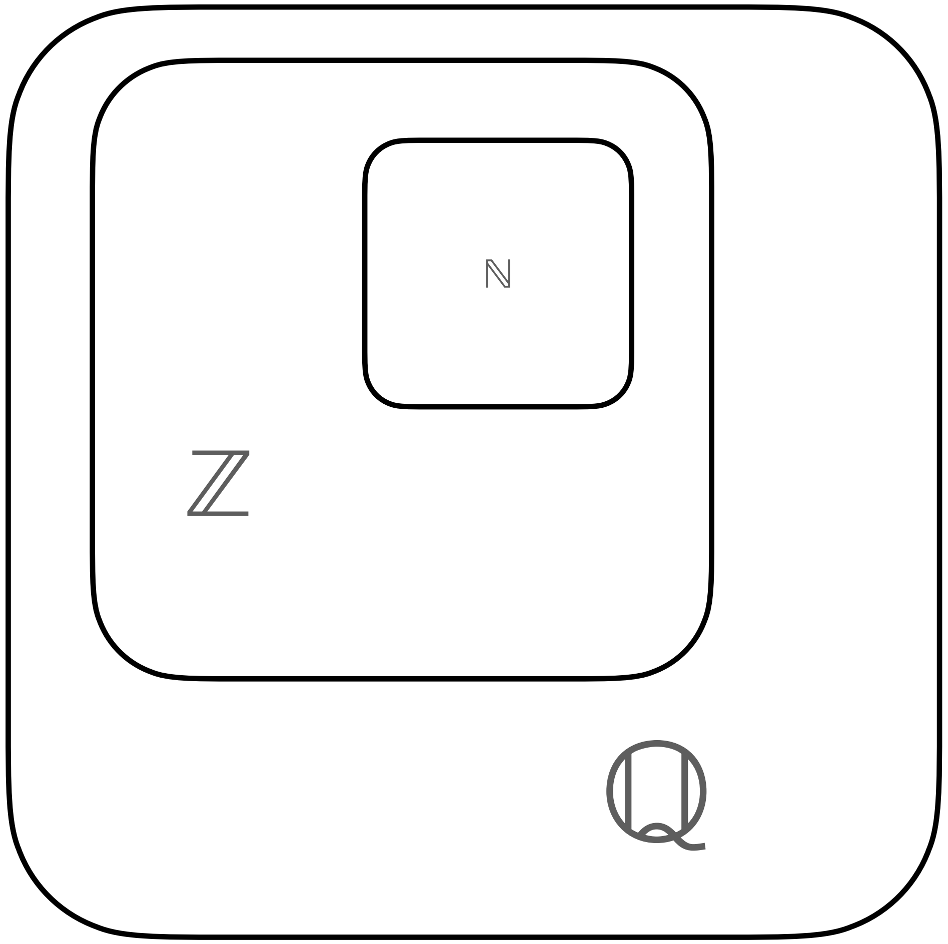

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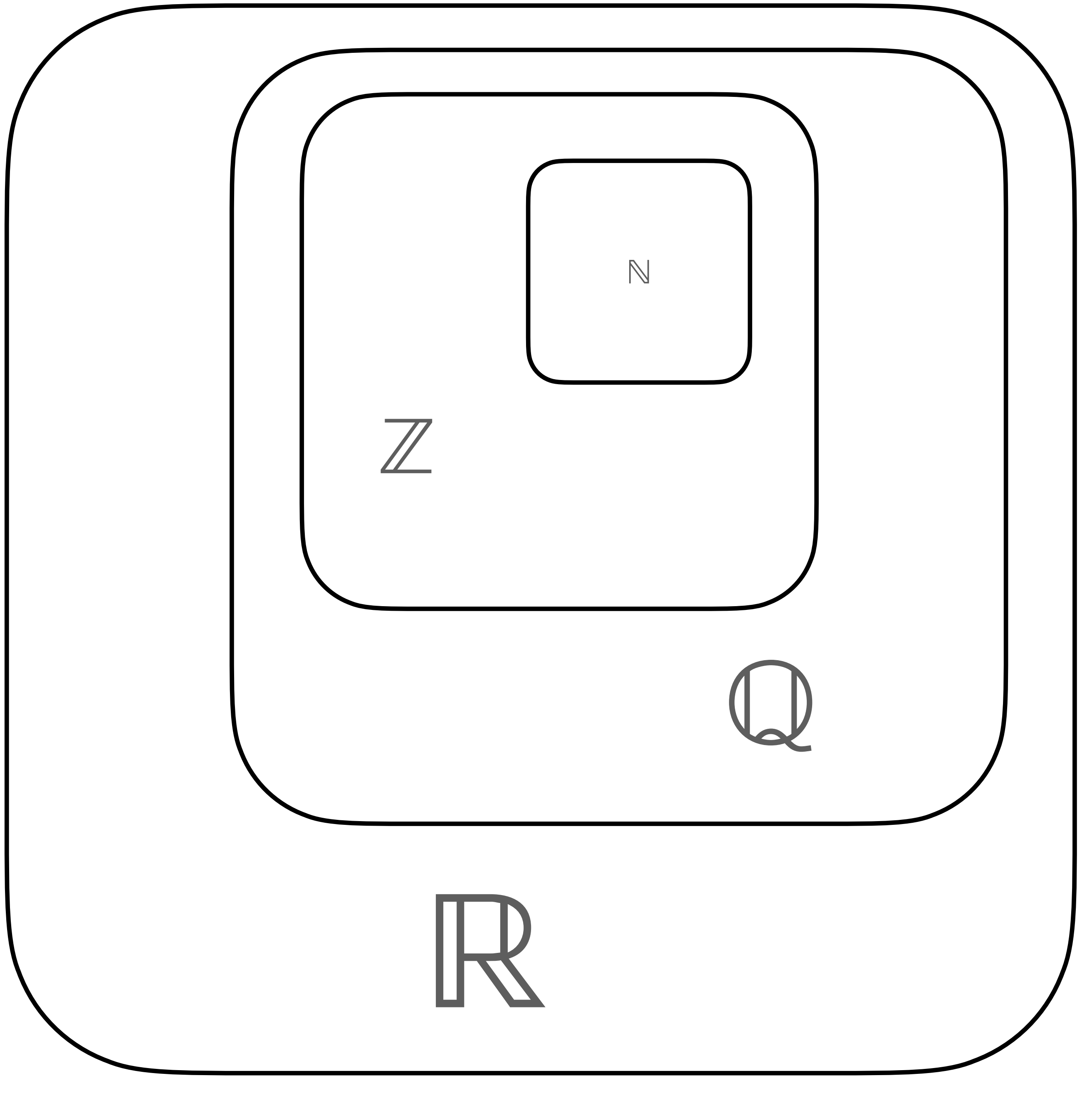
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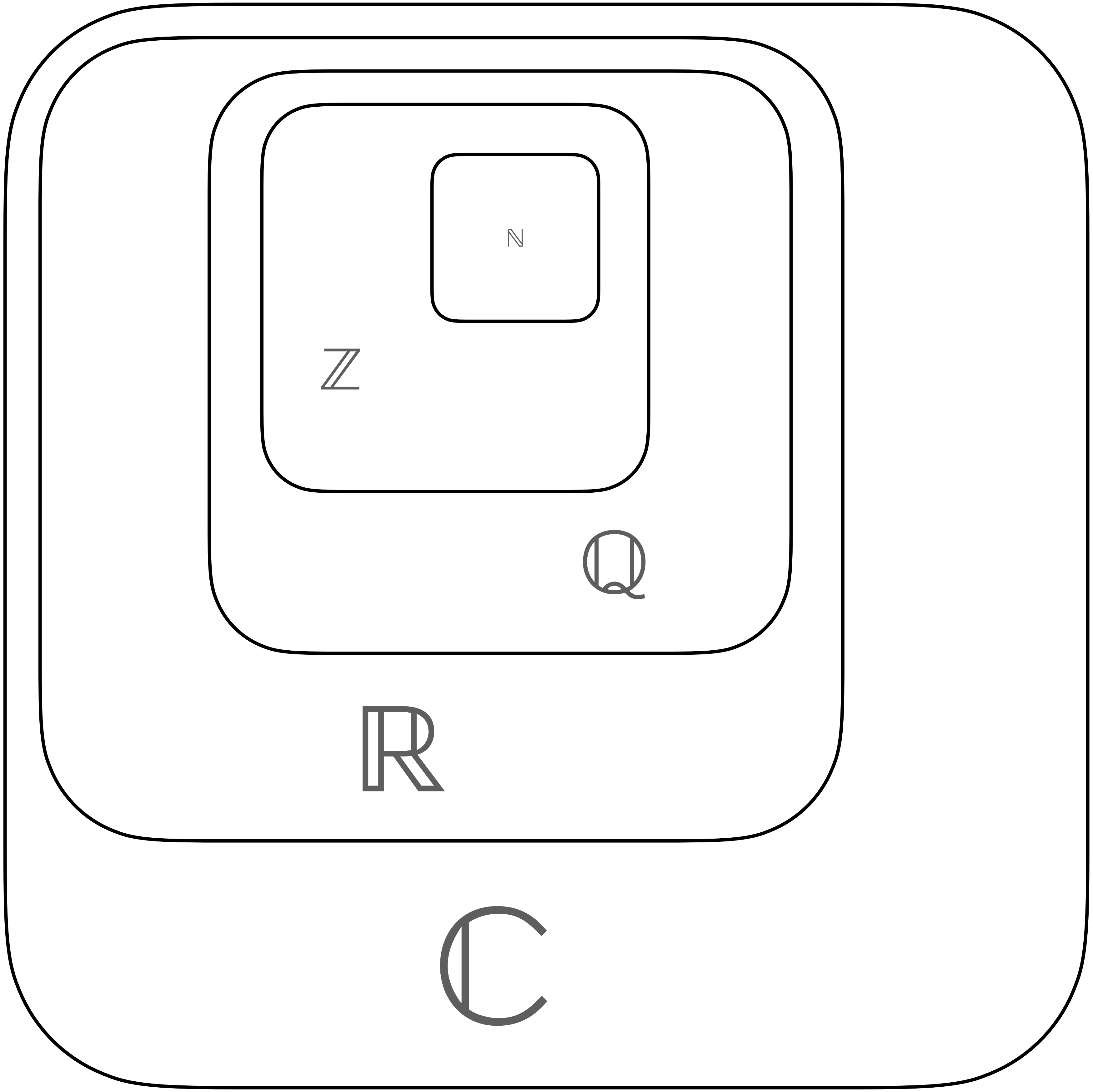
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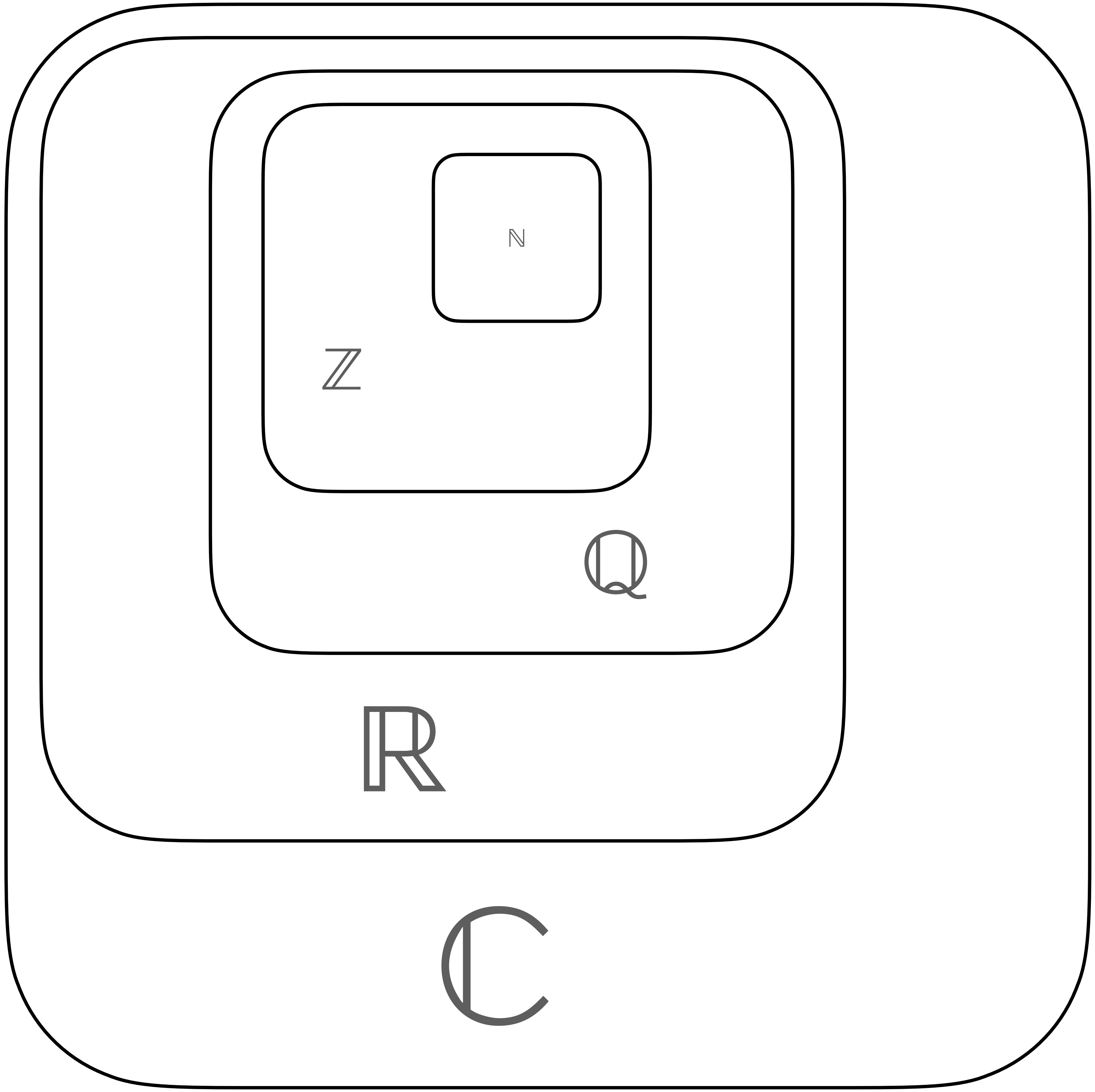

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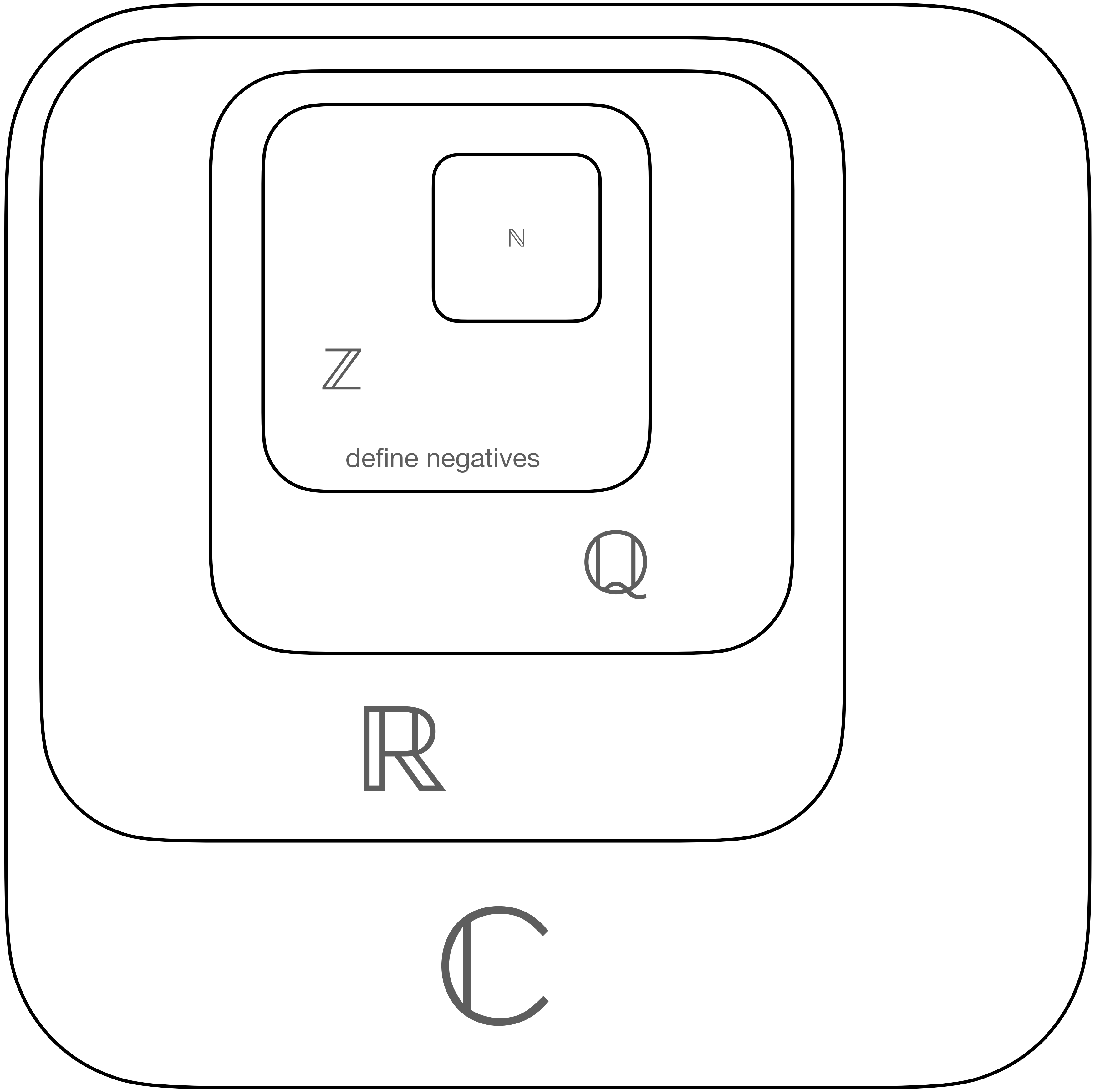

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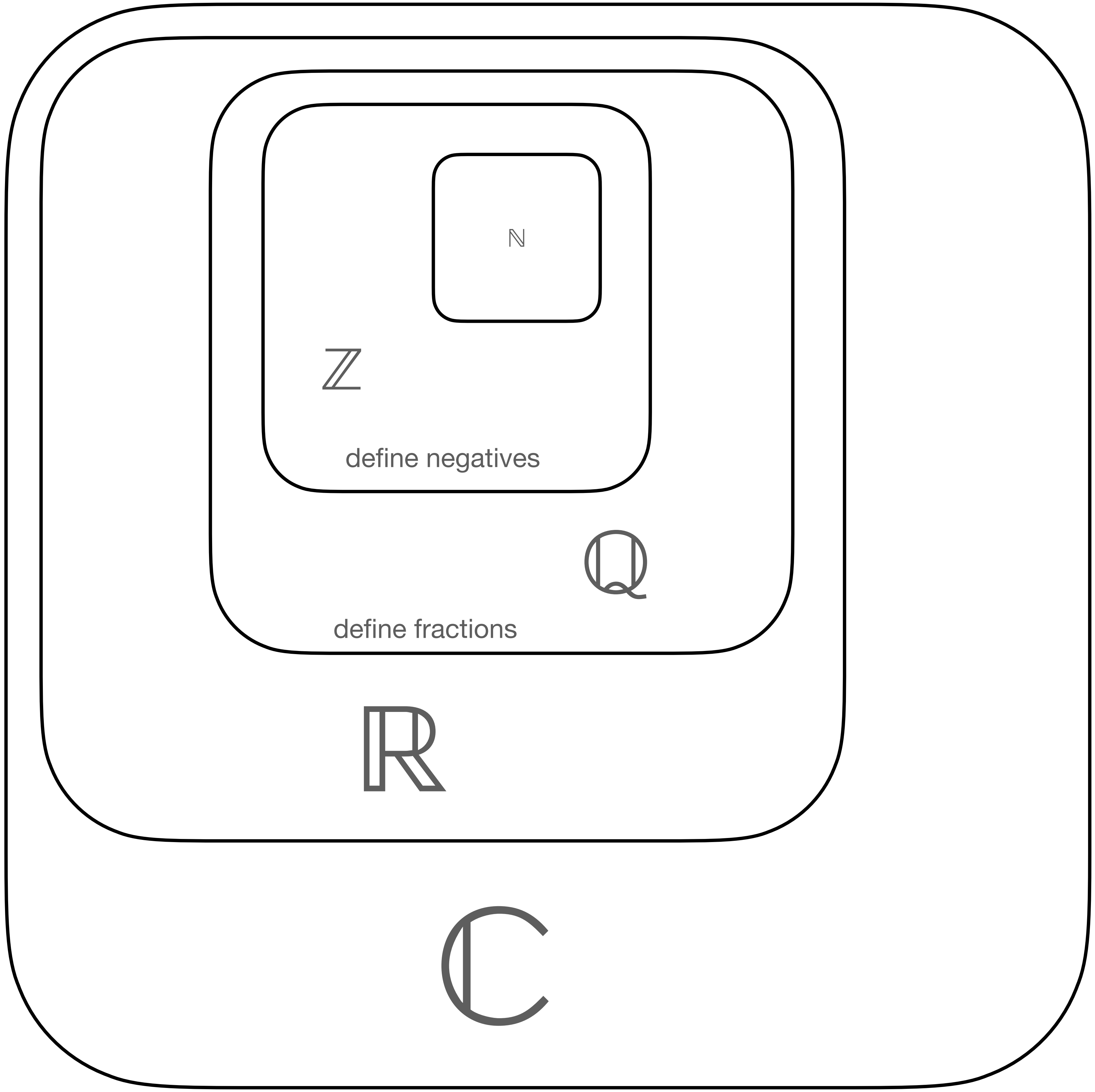

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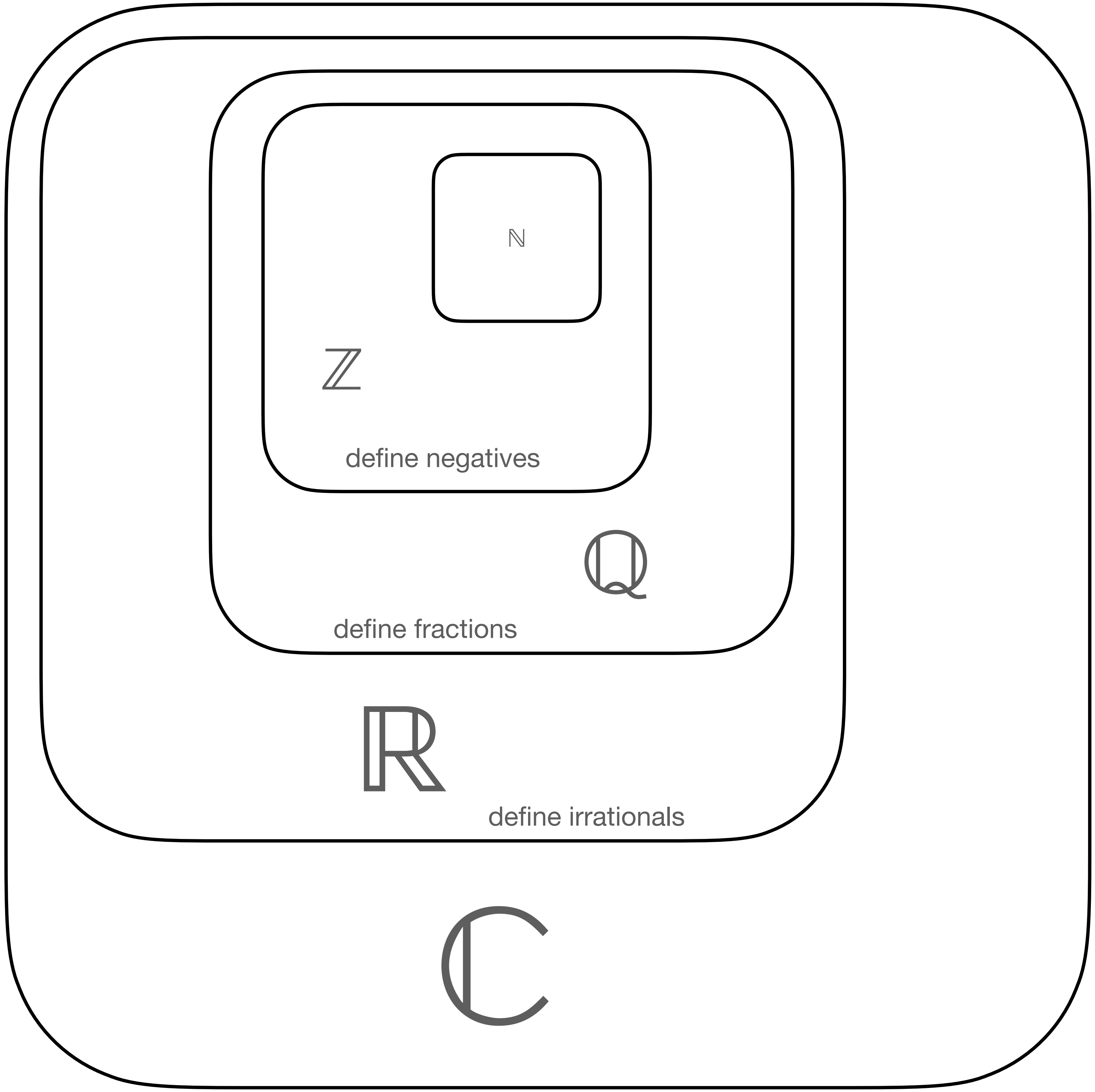

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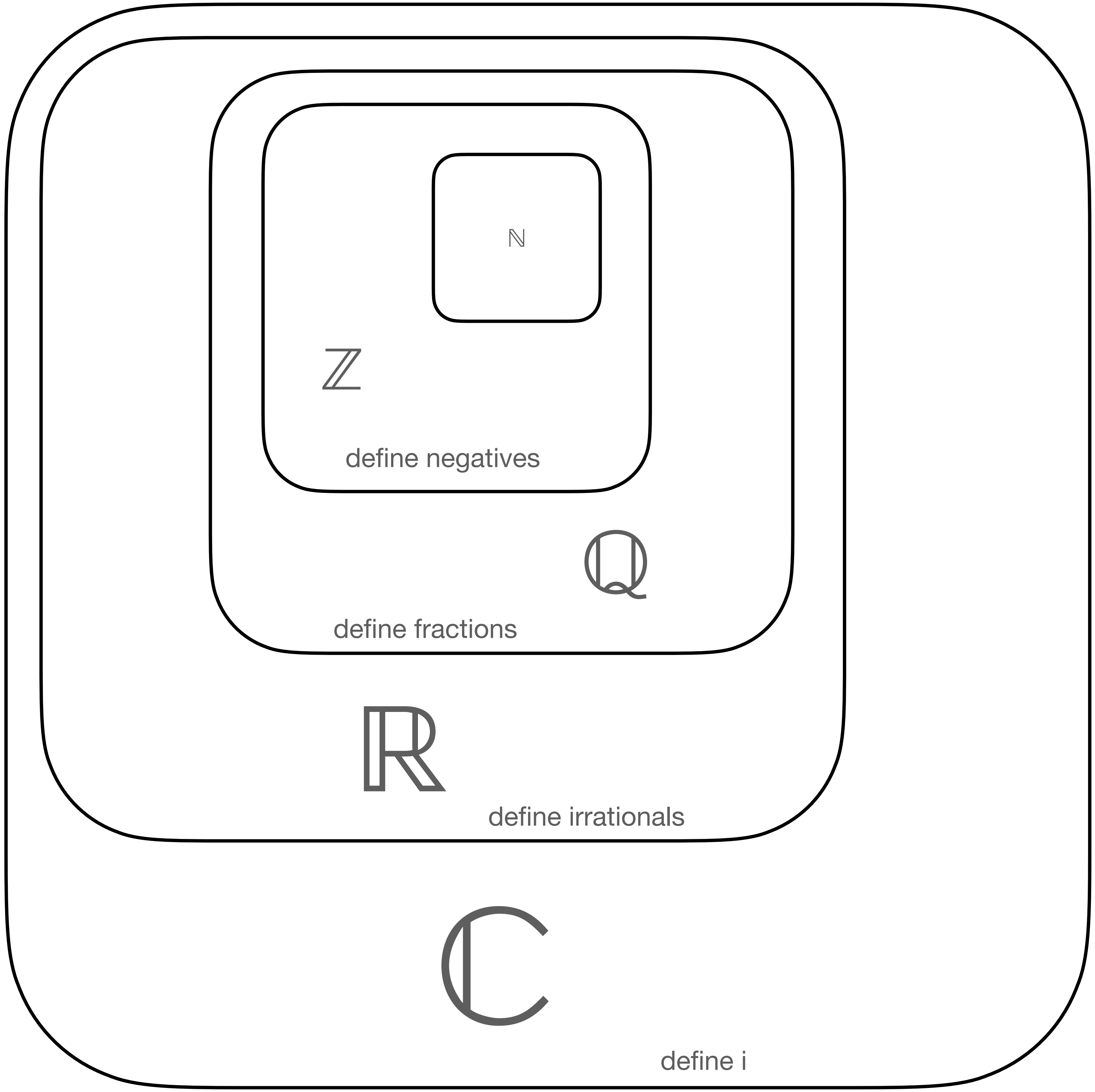

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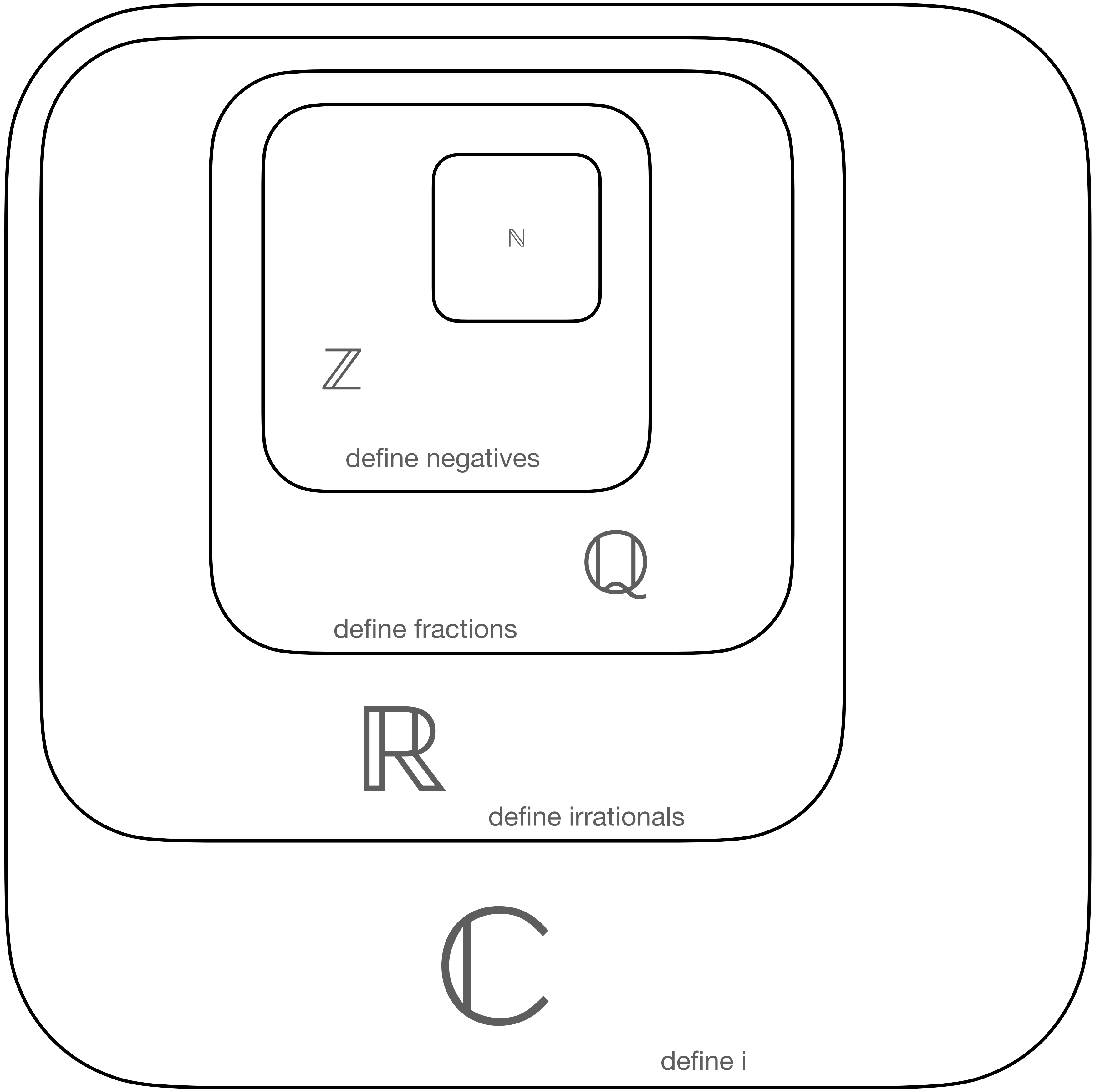

(this is the final stop)

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This will, in a bit, translate nicely into a cartesian graph coordinate

Add:

$$(a + bi) + (g + hi)$$

$$(a + g) + i(b + h)$$

Subtract

$$(a + bi) - (g + hi)$$

$$(a - g) + i(b - h)$$

Multiply

$$(a + bi) \times (g + hi)$$

$$ag + (ah)i + (bg)i + (bh)(i^2)$$

$$ag + (ah)i + (bg)i + (bh)(-1)$$

$$(ag - bh) + (ah + bg)i$$

Divide

$$(a + bi) \div (g + hi)$$

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This will, in a bit, translate nicely into a cartesian graph coordinate

But for now use it as as compact representation of complex numbers:

Add:	Subtract	Multiply	Divide
$(a+bi)+(g+hi)$	$(a+bi)-(g+hi)$	$(a+bi)\times(g+hi)$	$(a+bi)\div(g+hi)$
$(a+g)+i(b+h)$	$(a-g)+i(b-h)$	$ag+(ah)i+(bg)i+(bh)(i^2)$	
		$ag+(ah)i+(bg)i+(bh)(-1)$	
		$(ag-bh)+(ah+bg)i$	

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(a,b)*(g,h) = (ag-bh, ah+bg)

Add:	Subtract	Multiply	Divide
$(a+bi)+(g+hi)$	$(a+bi)-(g+hi)$	$(a+bi)\times(g+hi)$	$(a+bi)\div(g+hi)$
$(a+g)+i(b+h)$	$(a-g)+i(b-h)$	$ag+(ah)i+(bg)i+(bh)(i^2)$	
		$ag+(ah)i+(bg)i+(bh)(-1)$	
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We have mental models for what our various operations are

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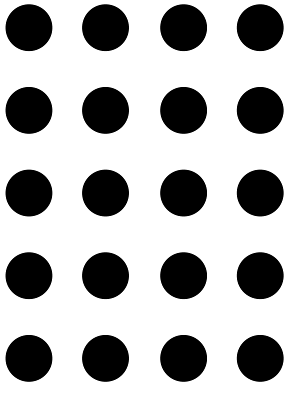
$\mathbb{N} : 4 \times 5$



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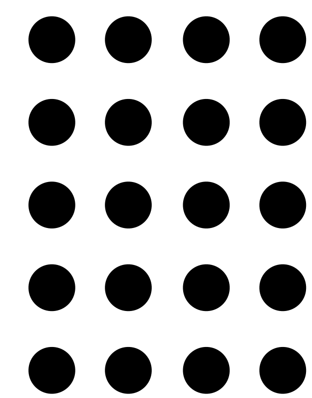
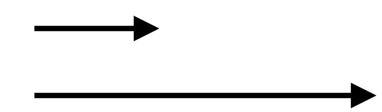


$\mathbb{R} : 3.42 * 234.3$

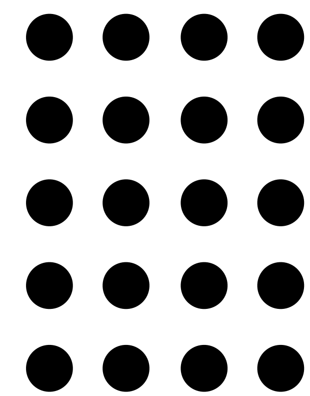


We have mental models for what our various operations are

$\mathbb{C} : (a + bi) \times (g + hi)$

(a,b)*(g,h) = (ag-bh, ah+bg)

$\mathbb{N} : 4 \times 5$



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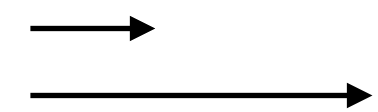

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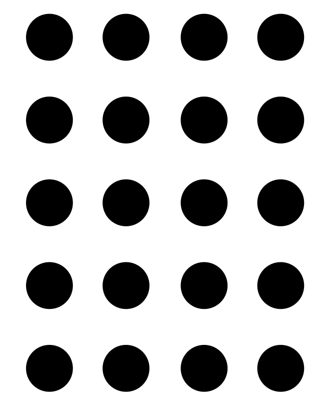

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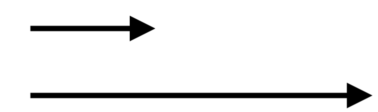
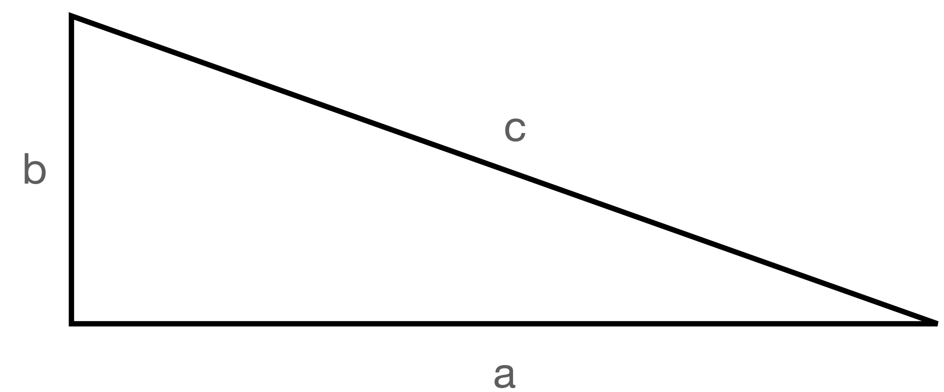
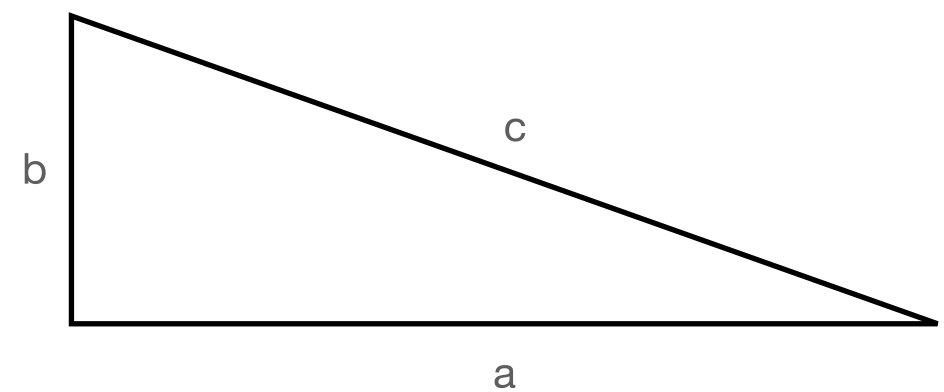


What is our mental model for “seeing” complex numbers multiplied?


c
b
a

In a right triangle, by definition



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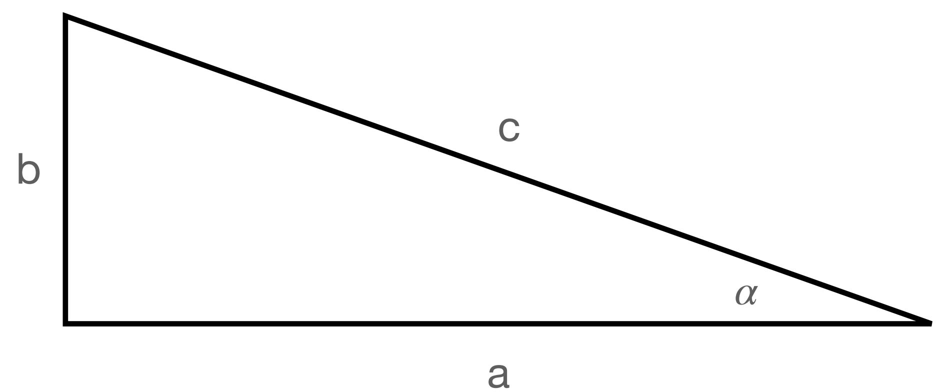
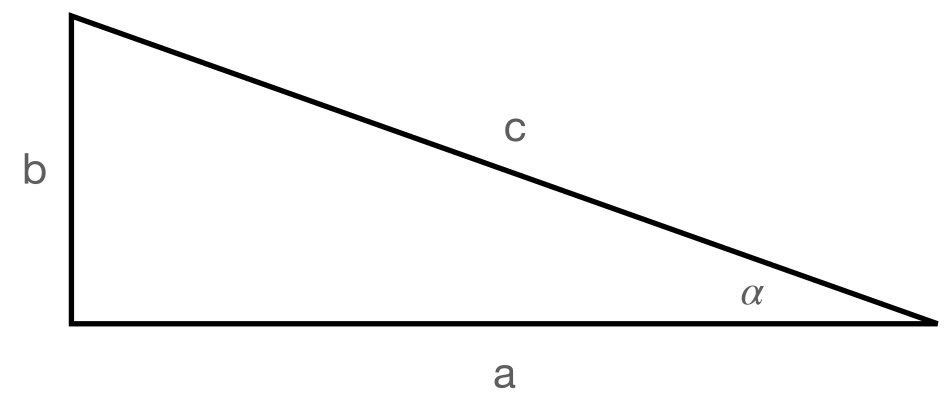


In a right triangle, by definition



$$sin(\alpha) = \frac{opp}{hyp} = \frac{b}{c}$$

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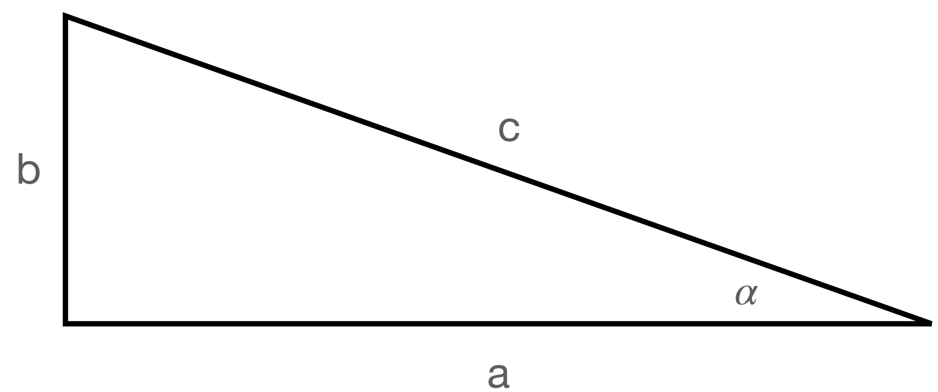

$$sin(\alpha) = \frac{opp}{hyp} = \frac{b}{c}$$

$$cos(\alpha) = \frac{adj}{hyp} = \frac{a}{c}$$

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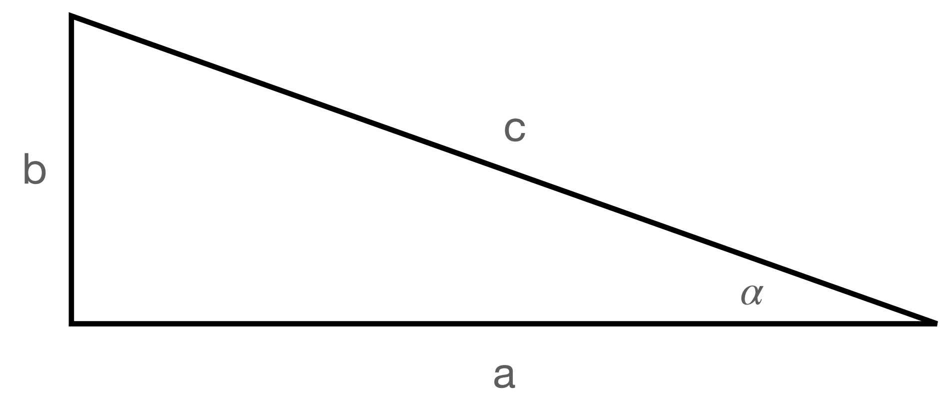


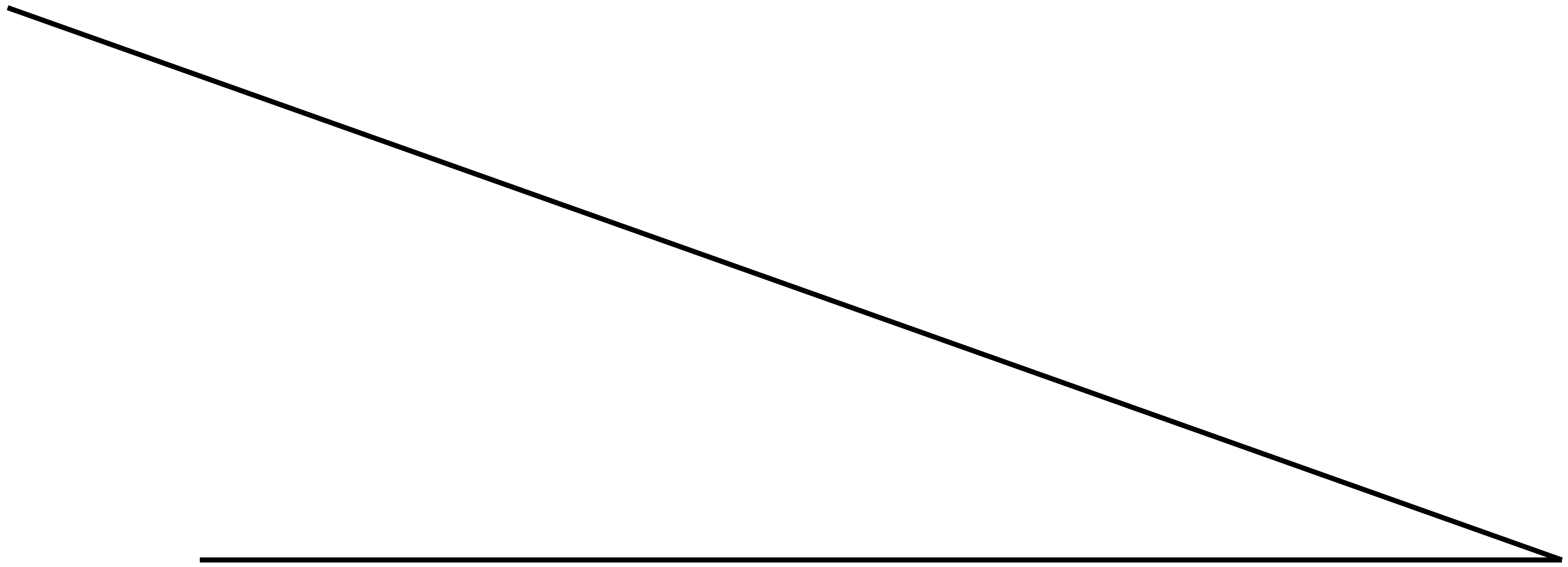
$$sin(\alpha) = \frac{opp}{hyp} = \frac{b}{c}$$

$$cos(\alpha) = \frac{adj}{hyp} = \frac{a}{c}$$

Notice these are ratios - no units. The size of the triangle is ultimately irrelevant and is normalized away (by c)

If I have 2 angles, can I build their combined properties from the single angles?

If I have 2 angles, can I build their combined properties from the single angles?



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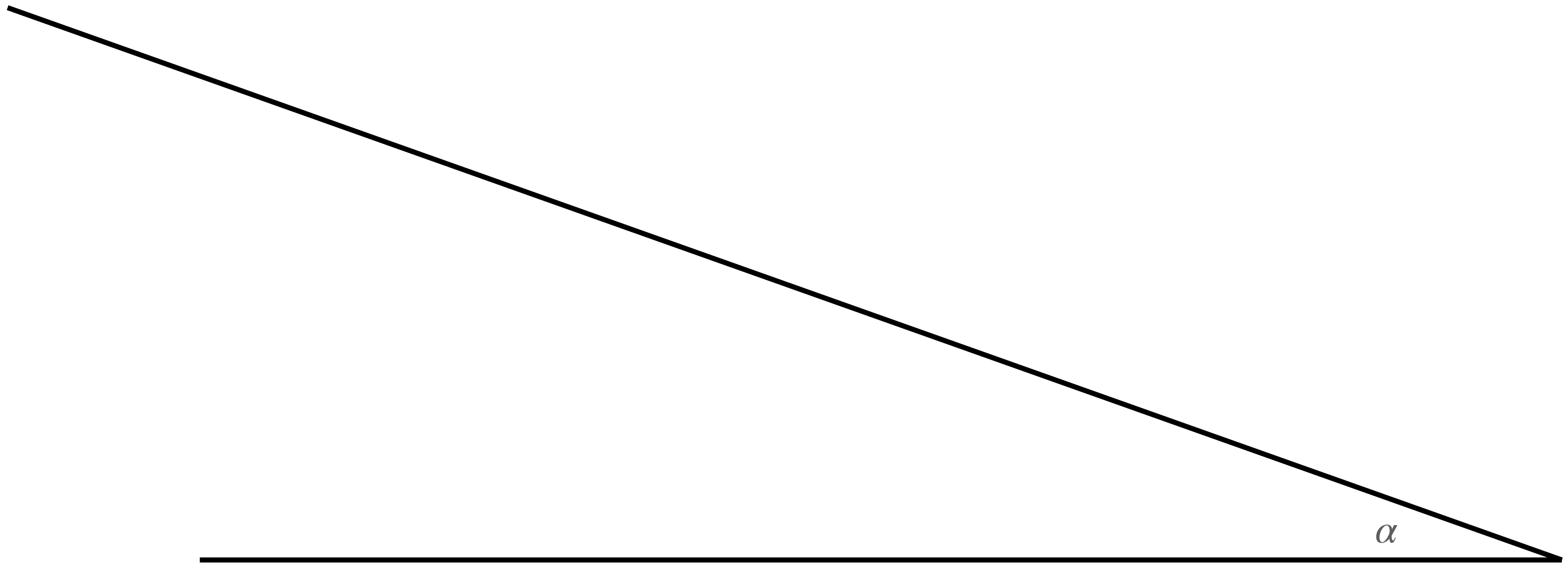

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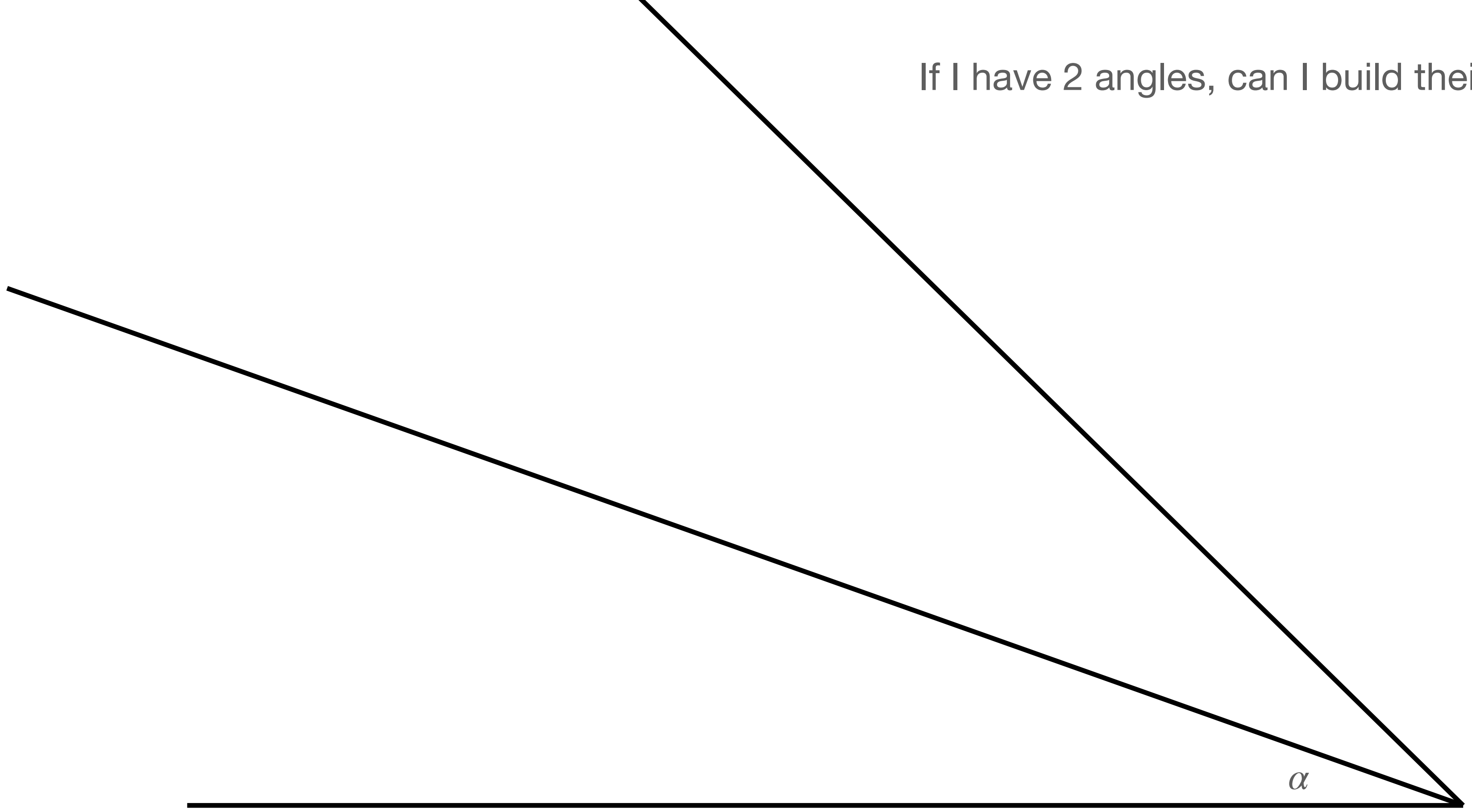

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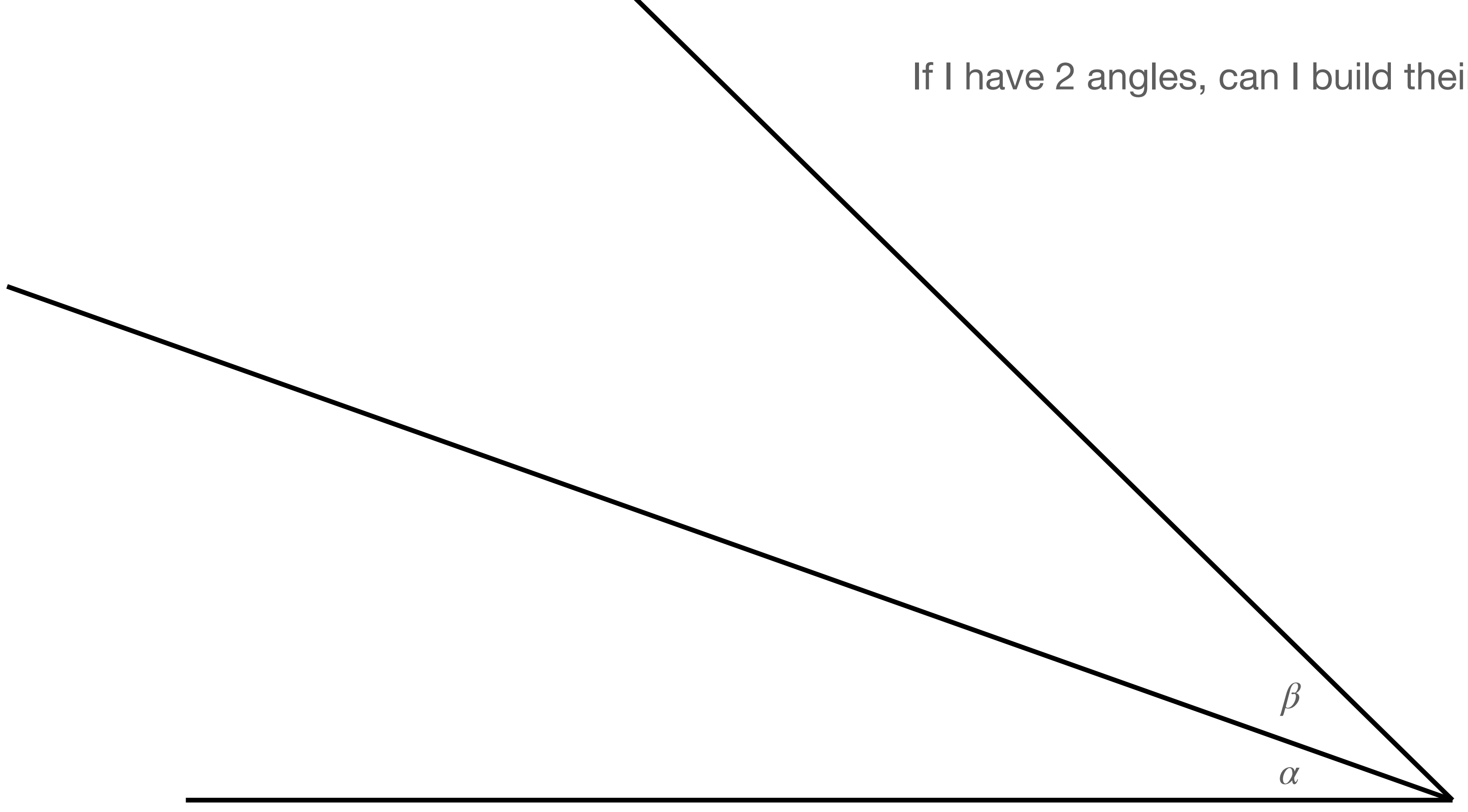

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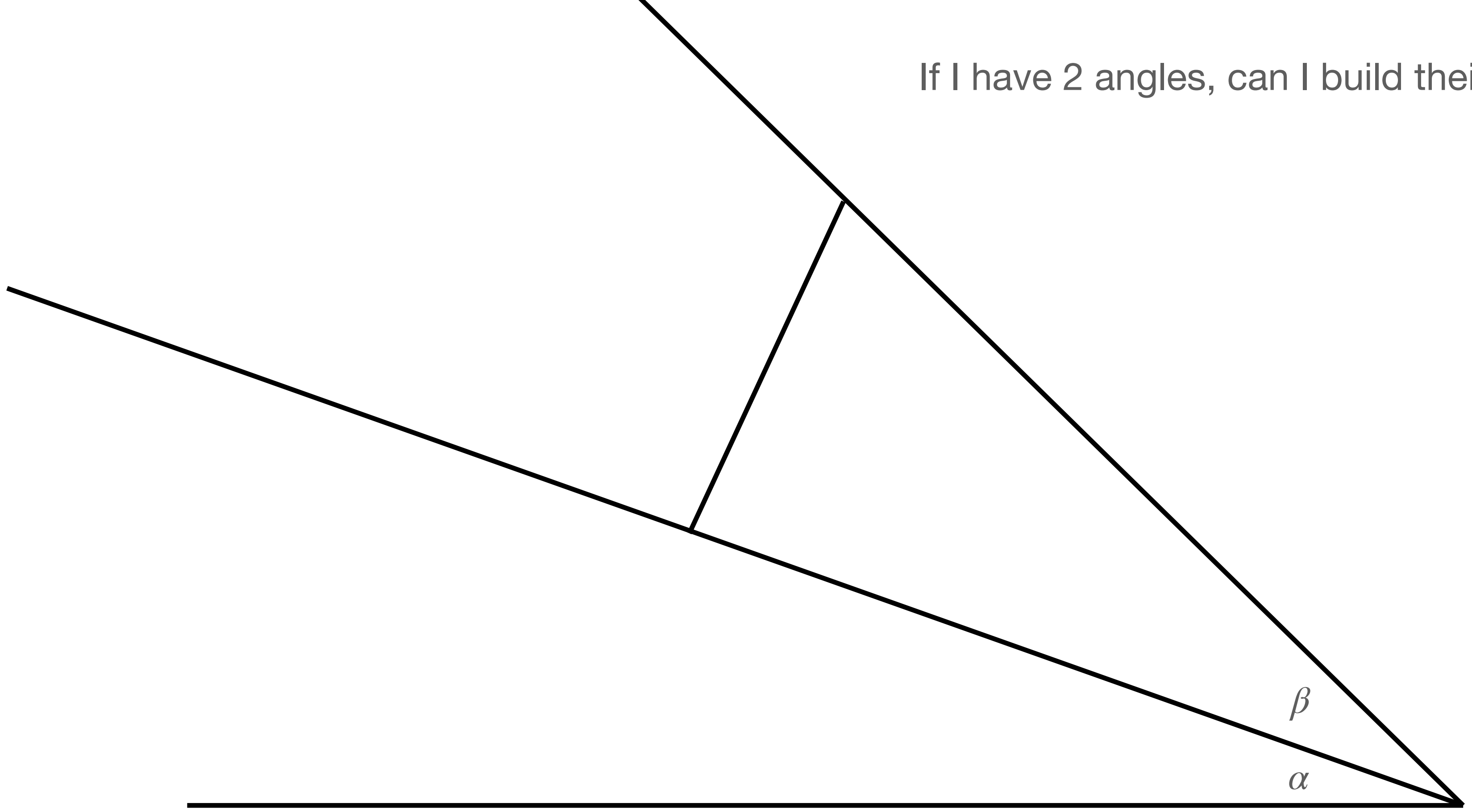

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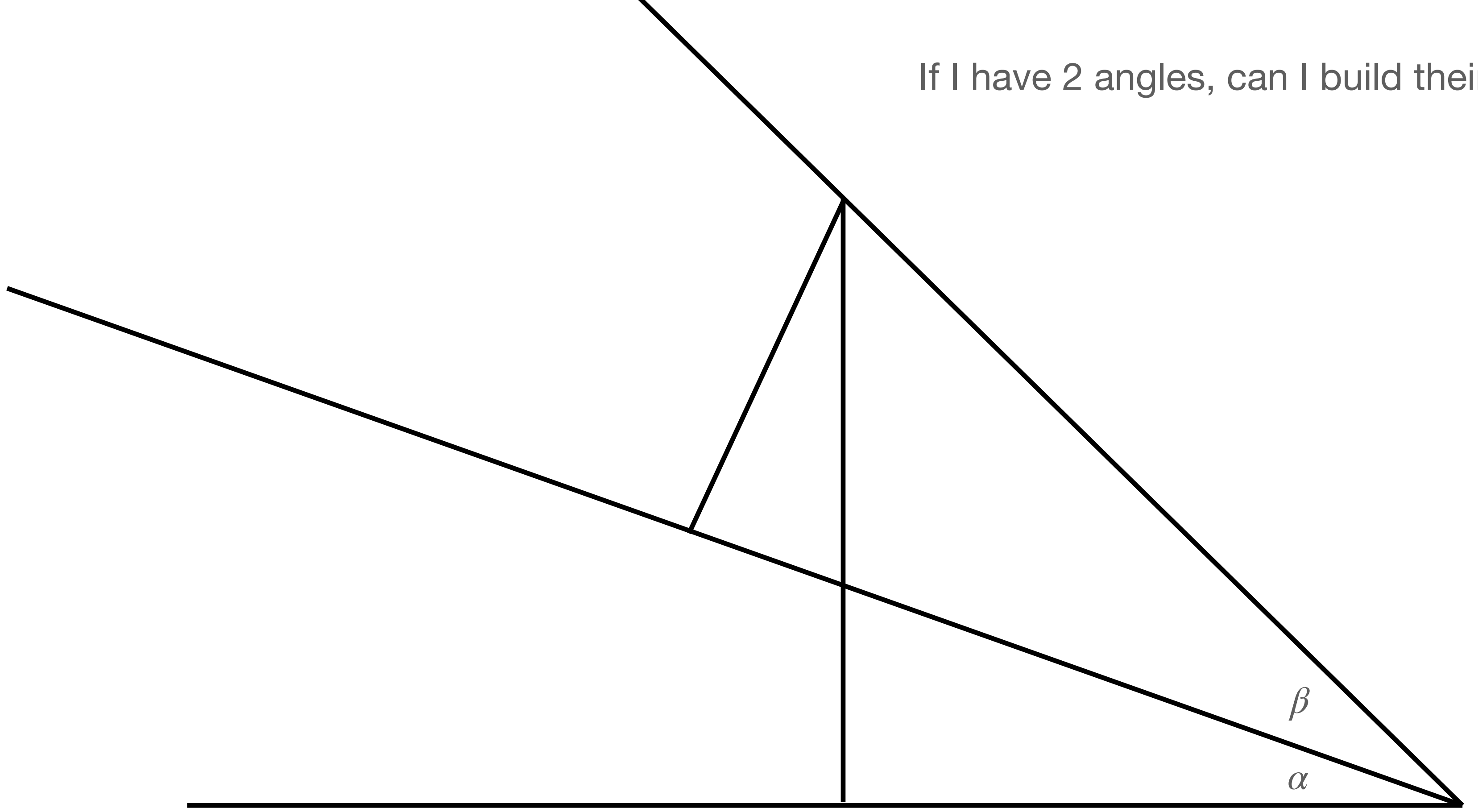

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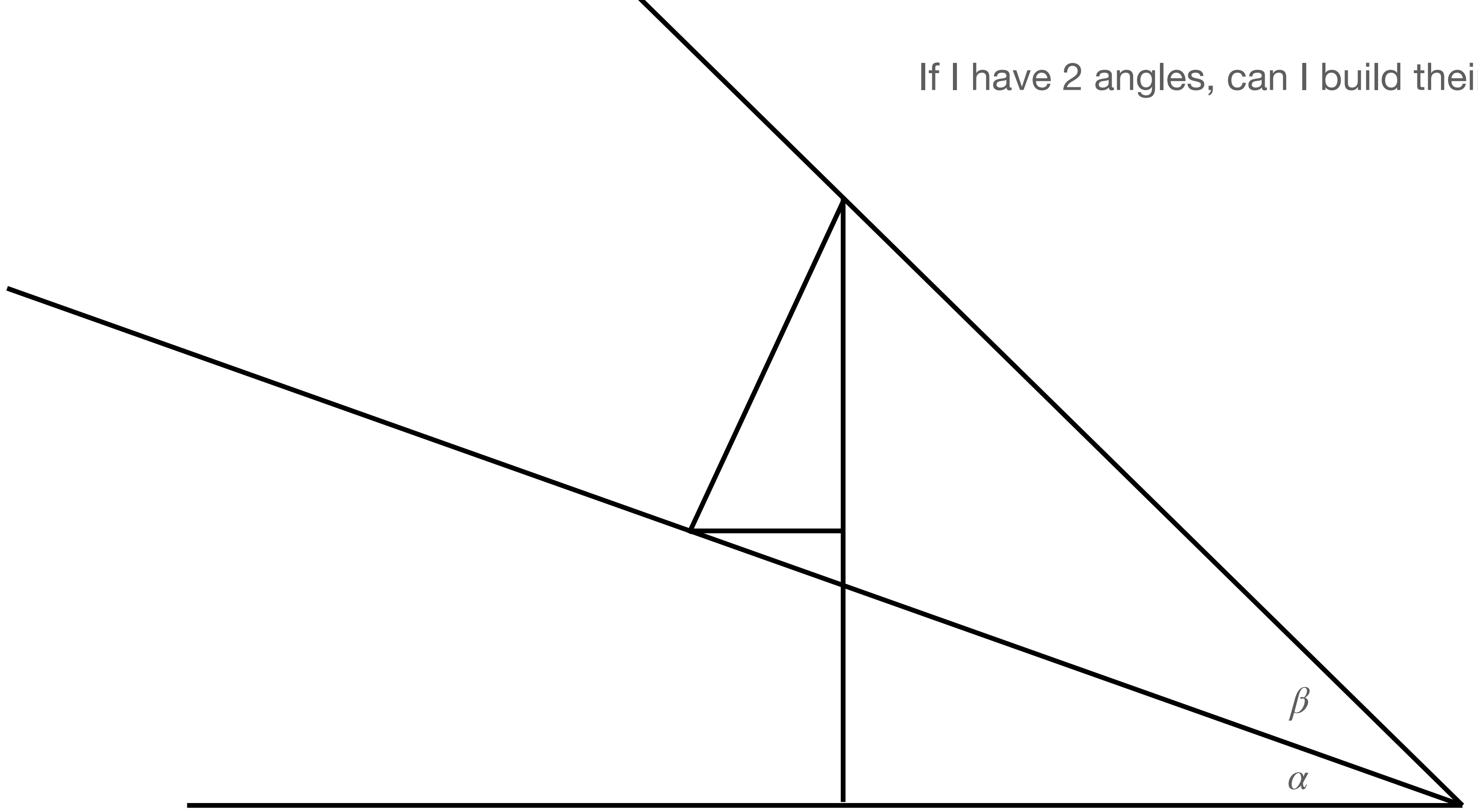

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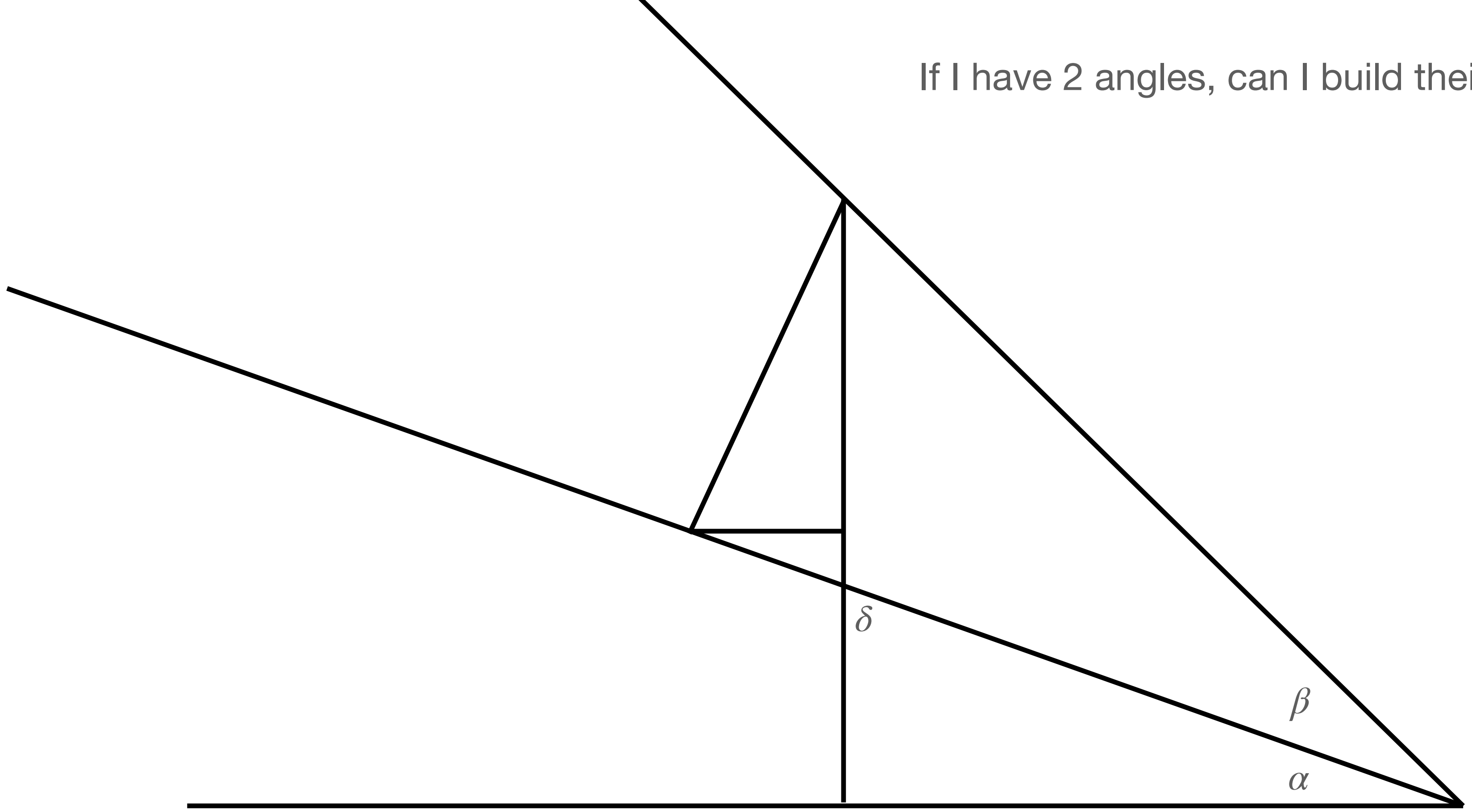

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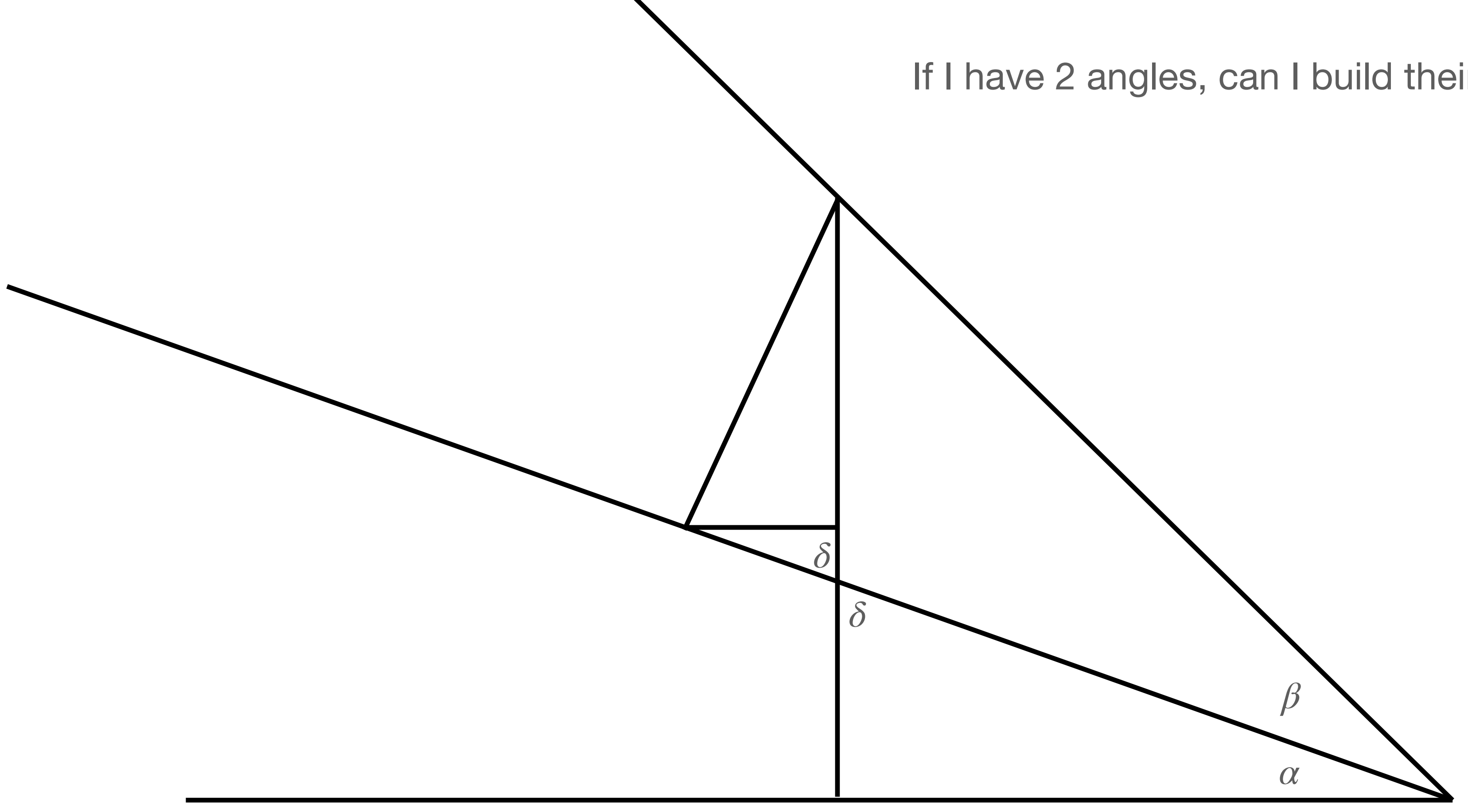

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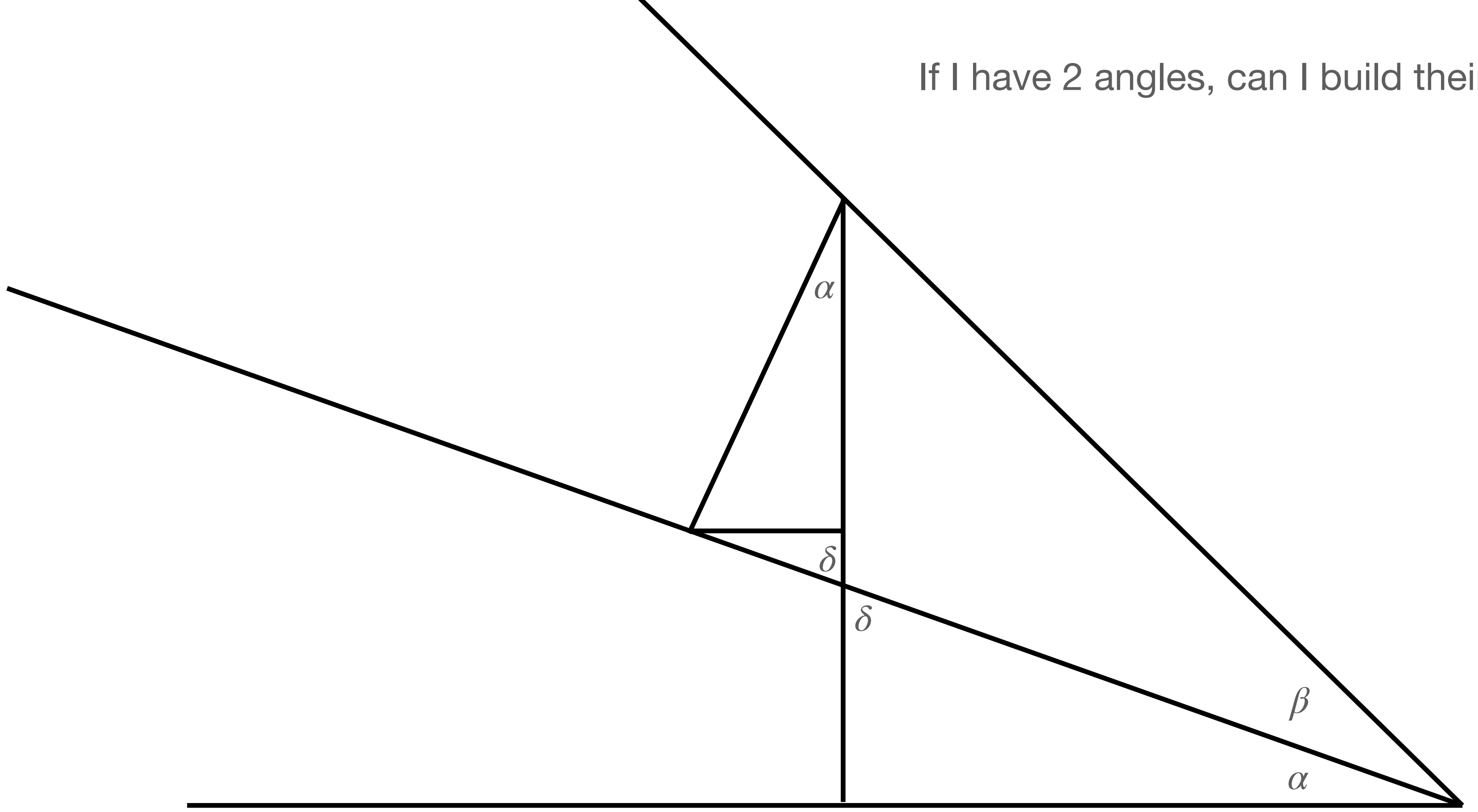

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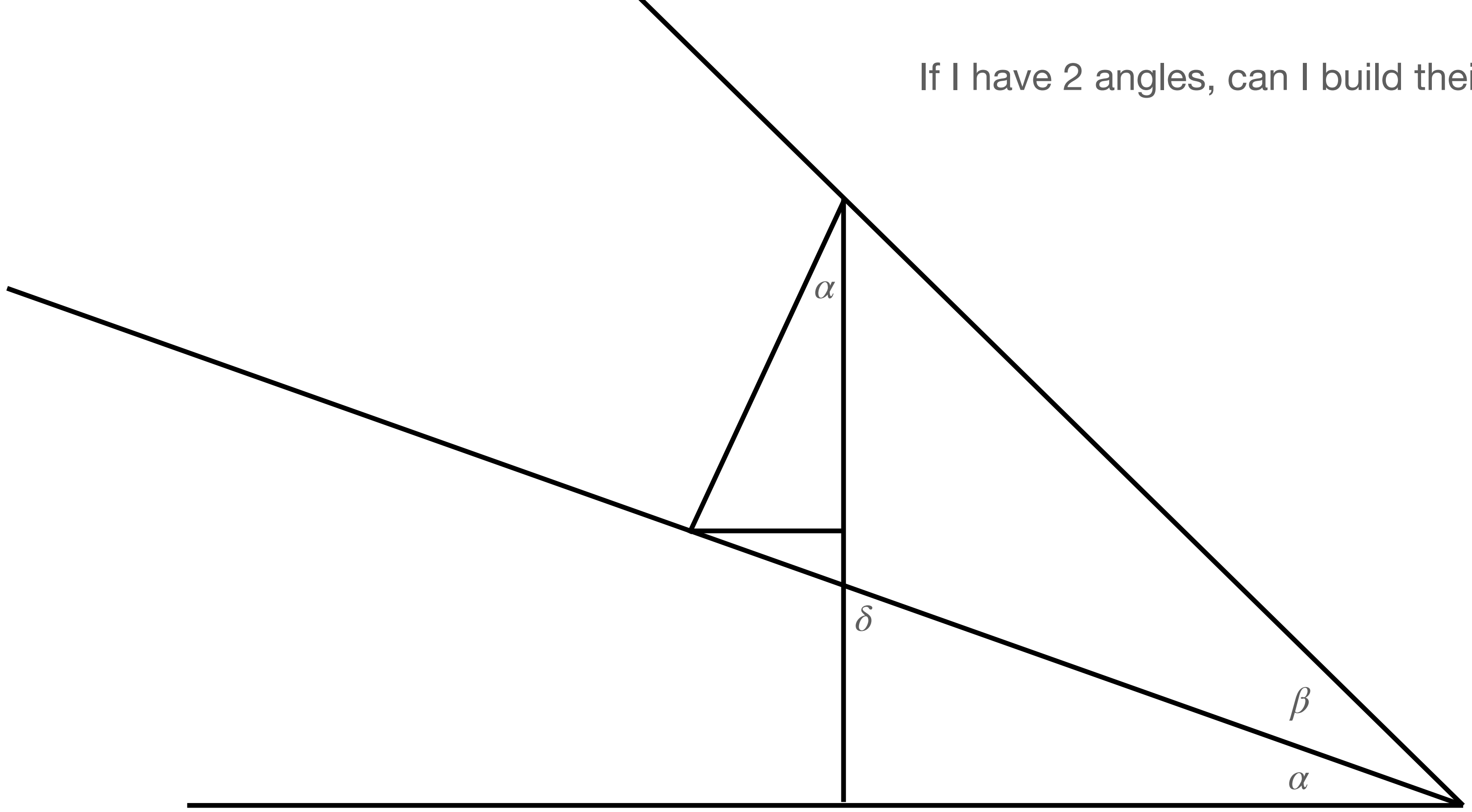

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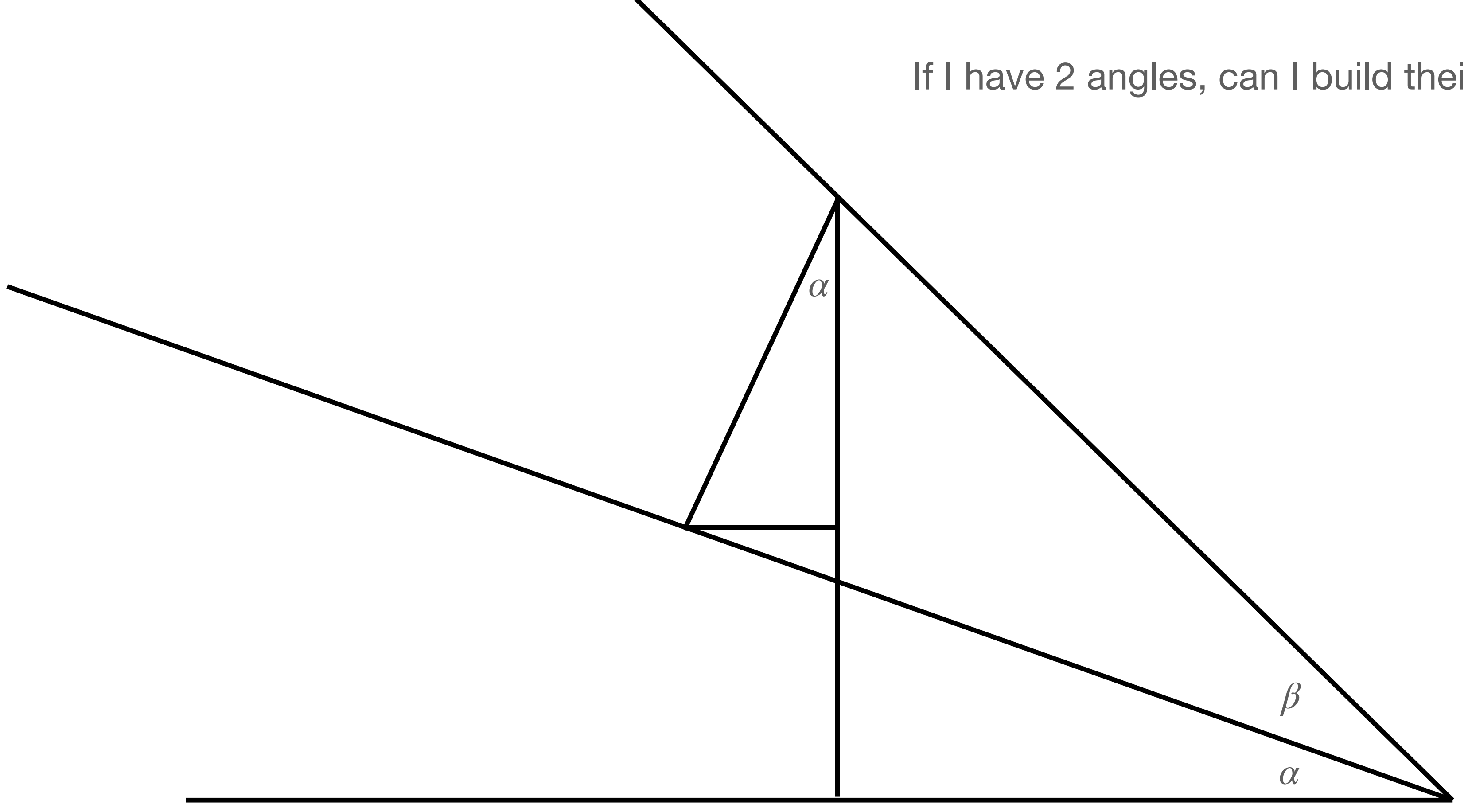

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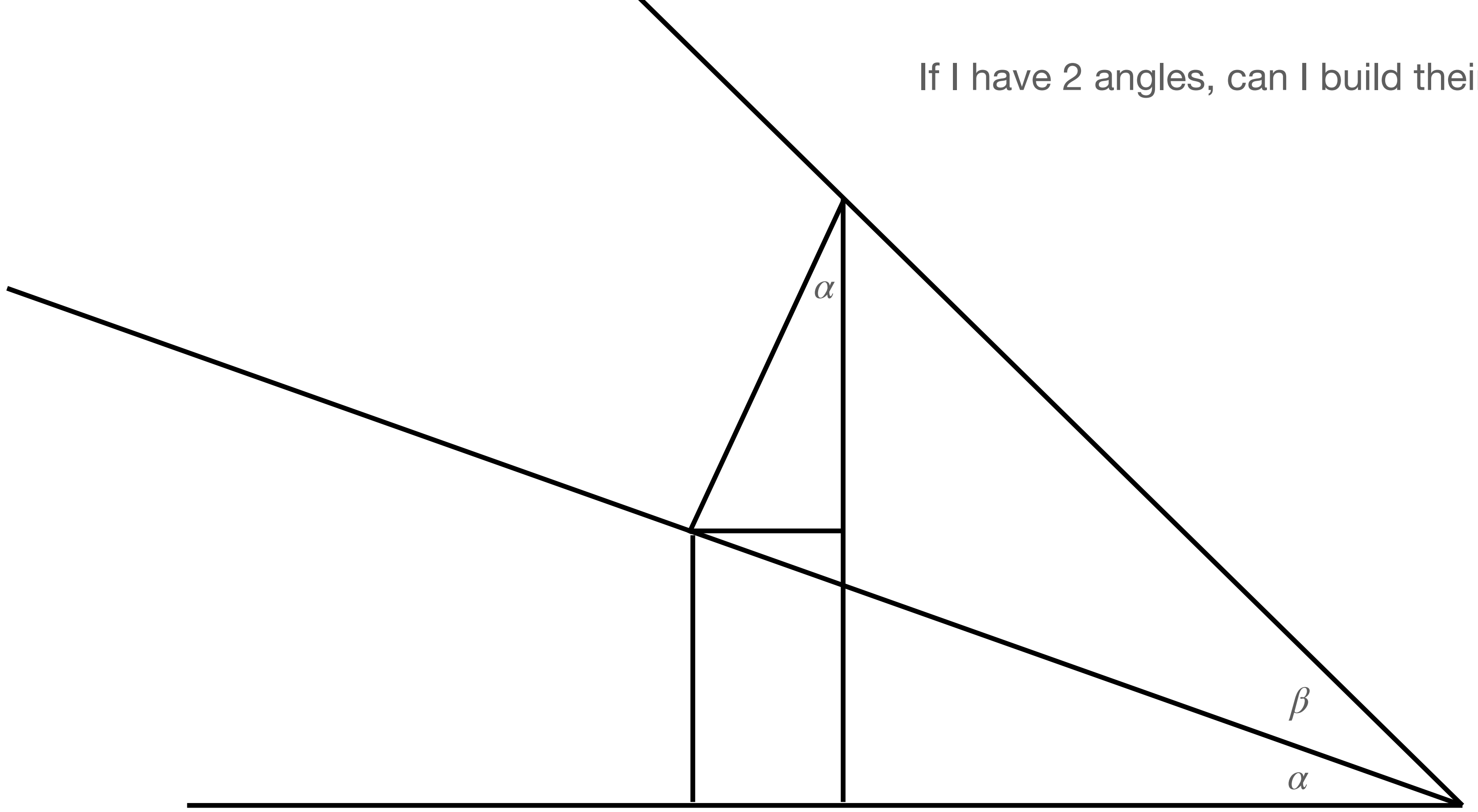

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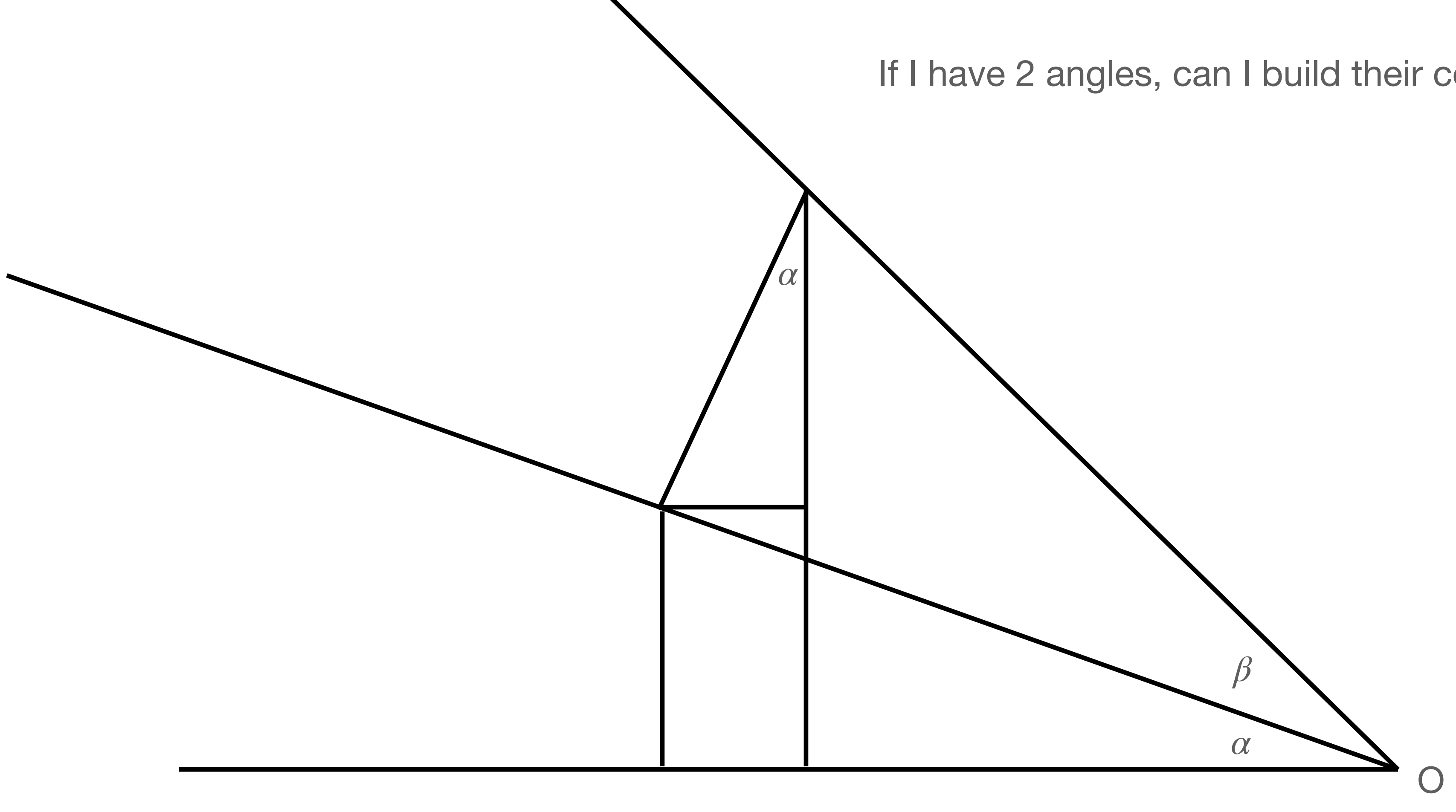

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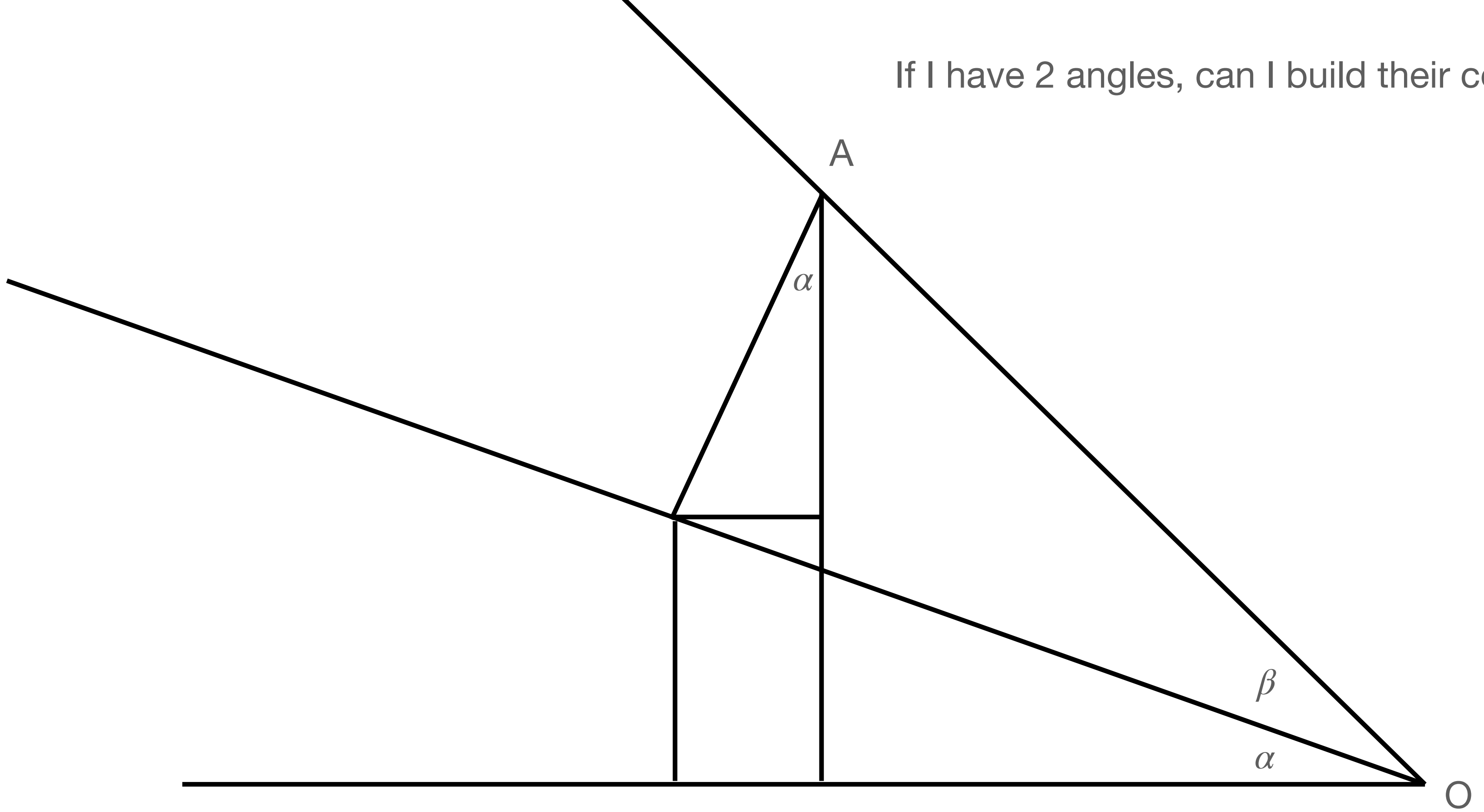

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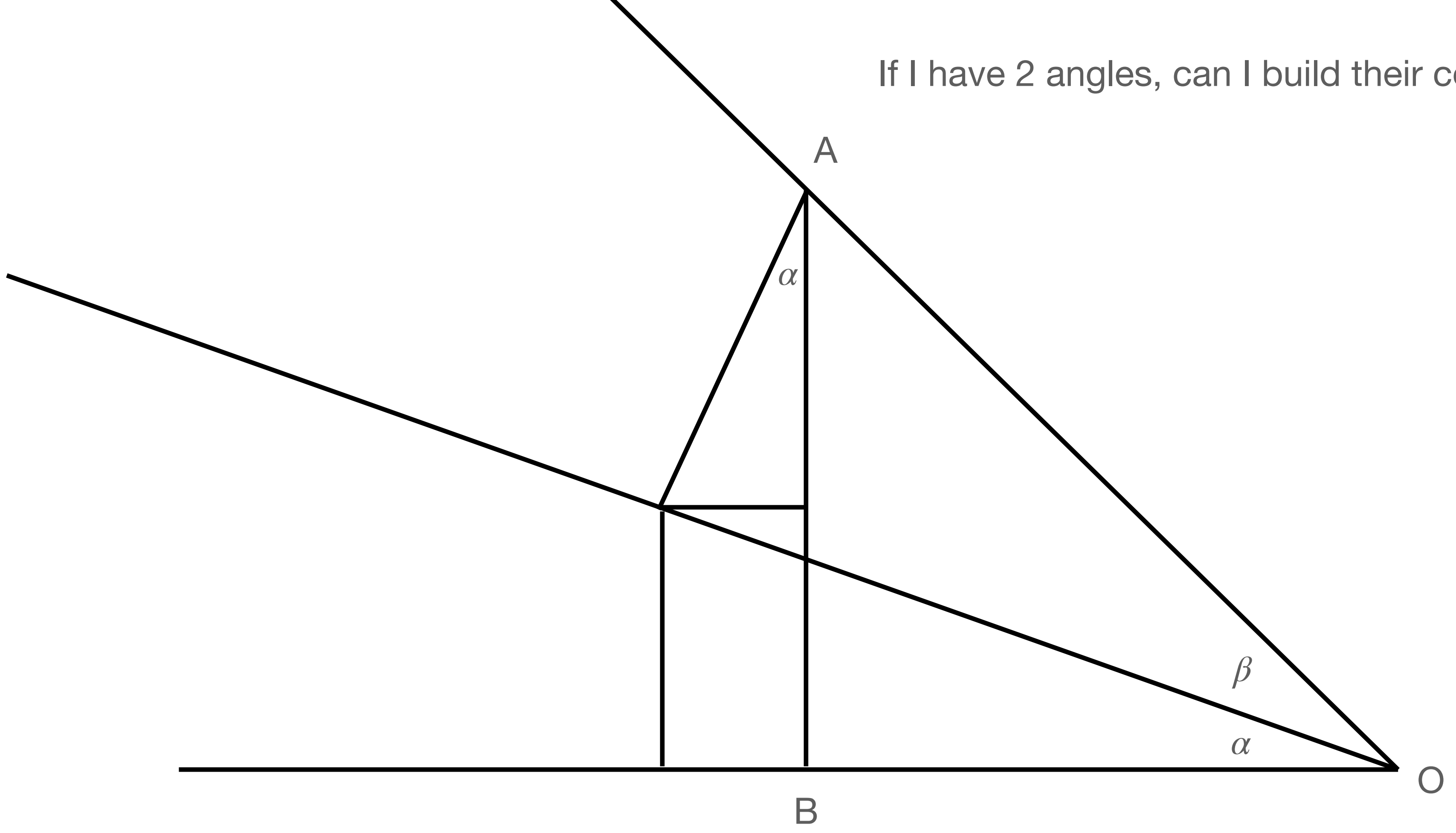

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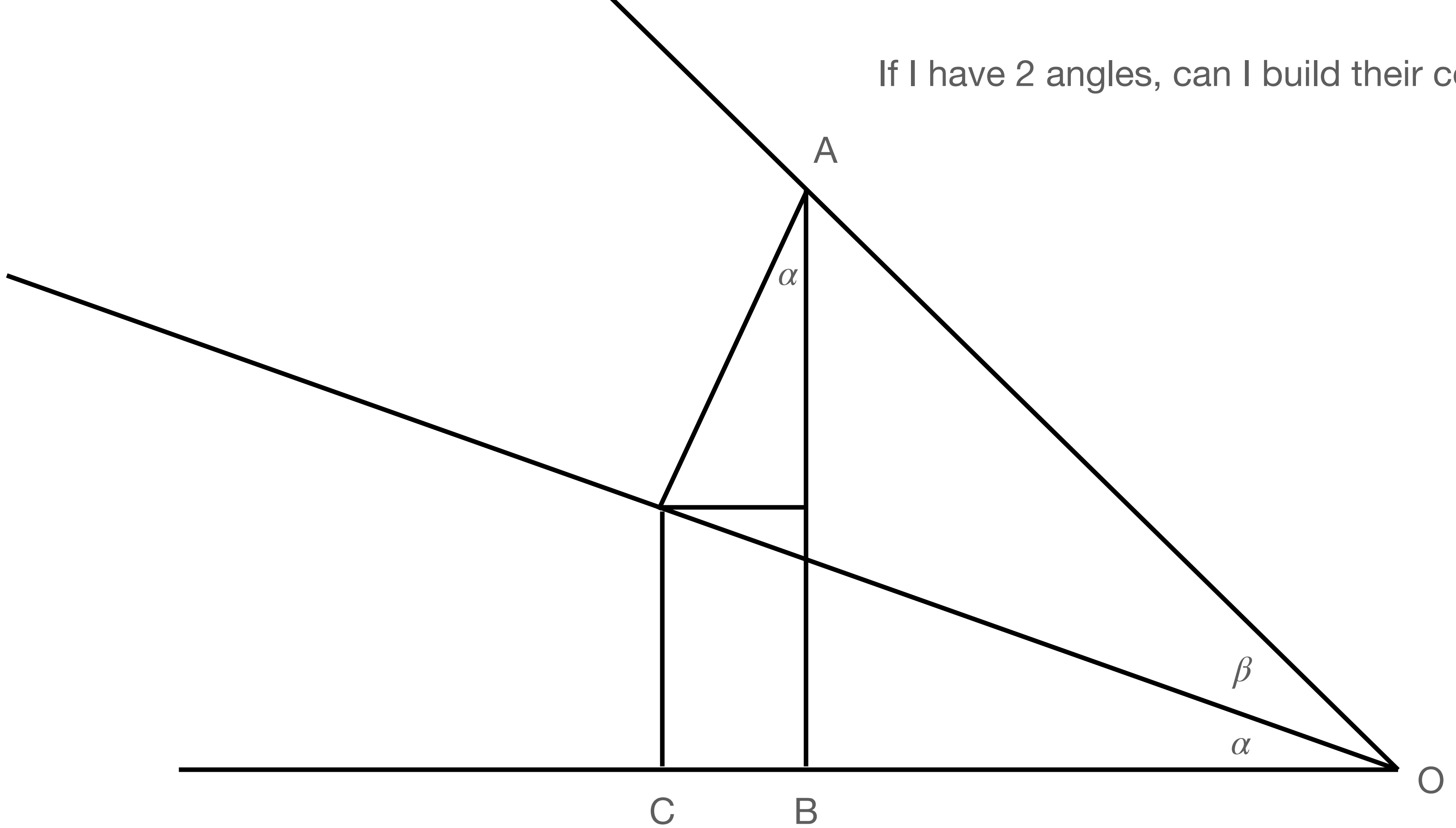

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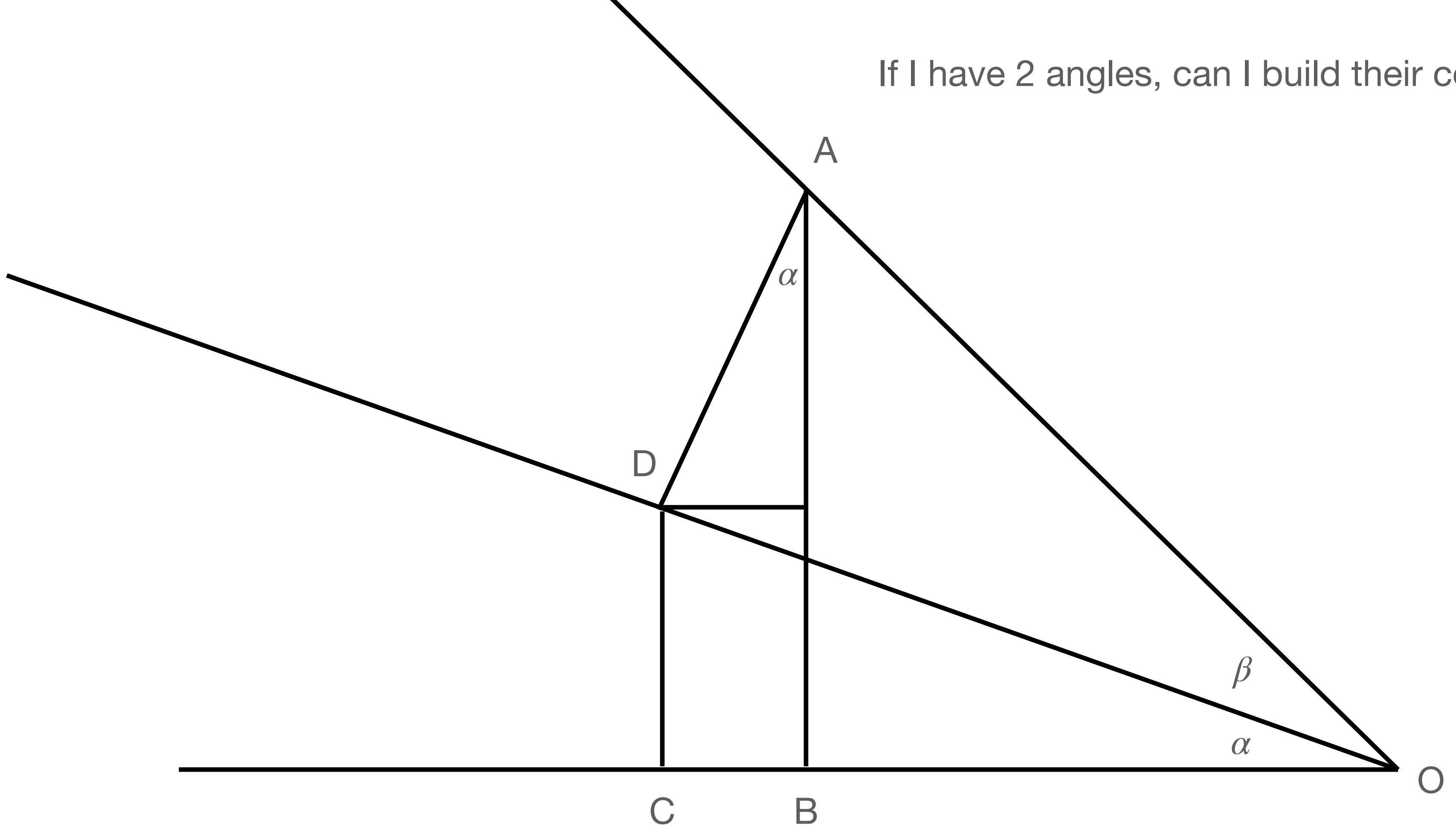

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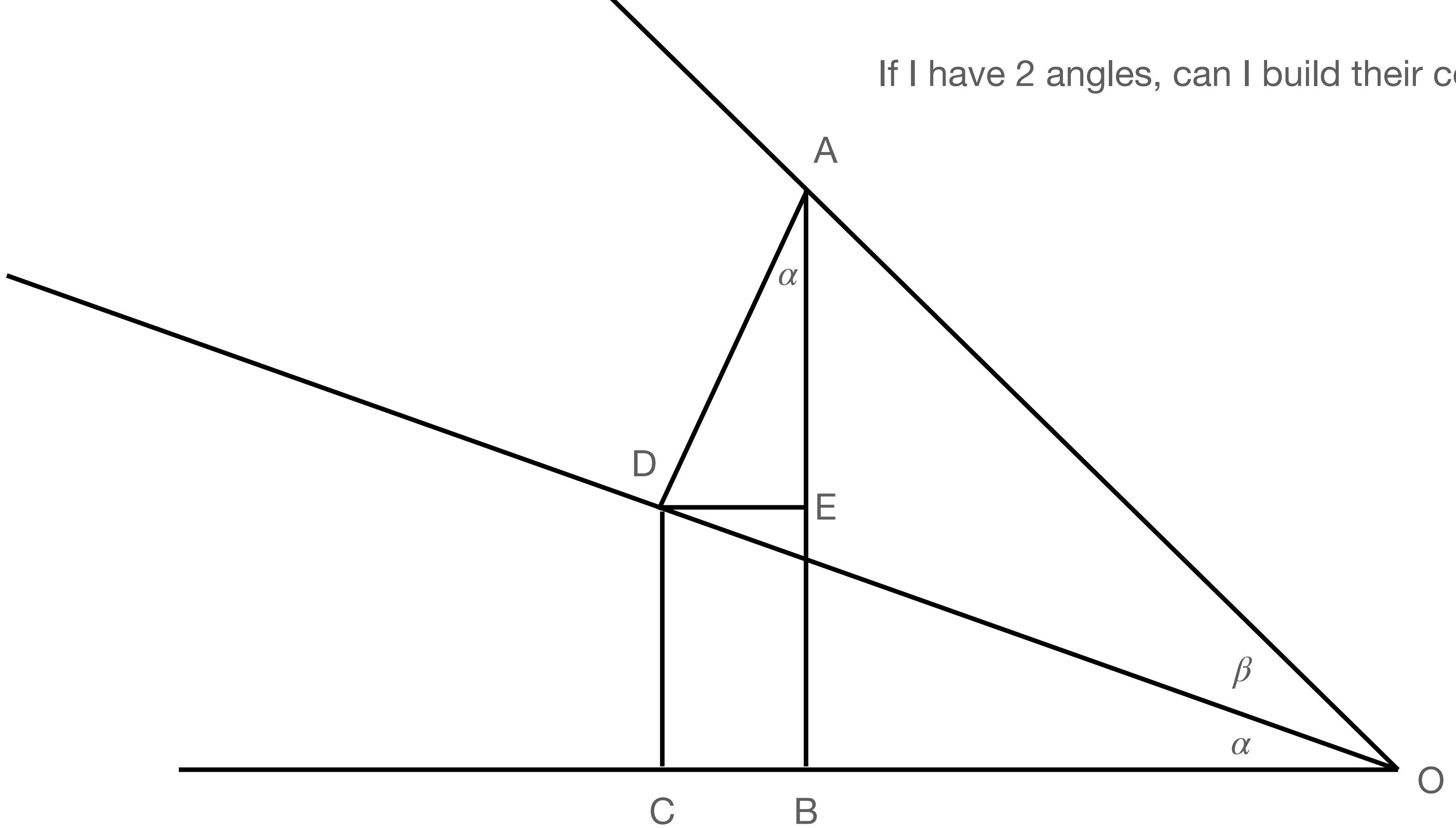

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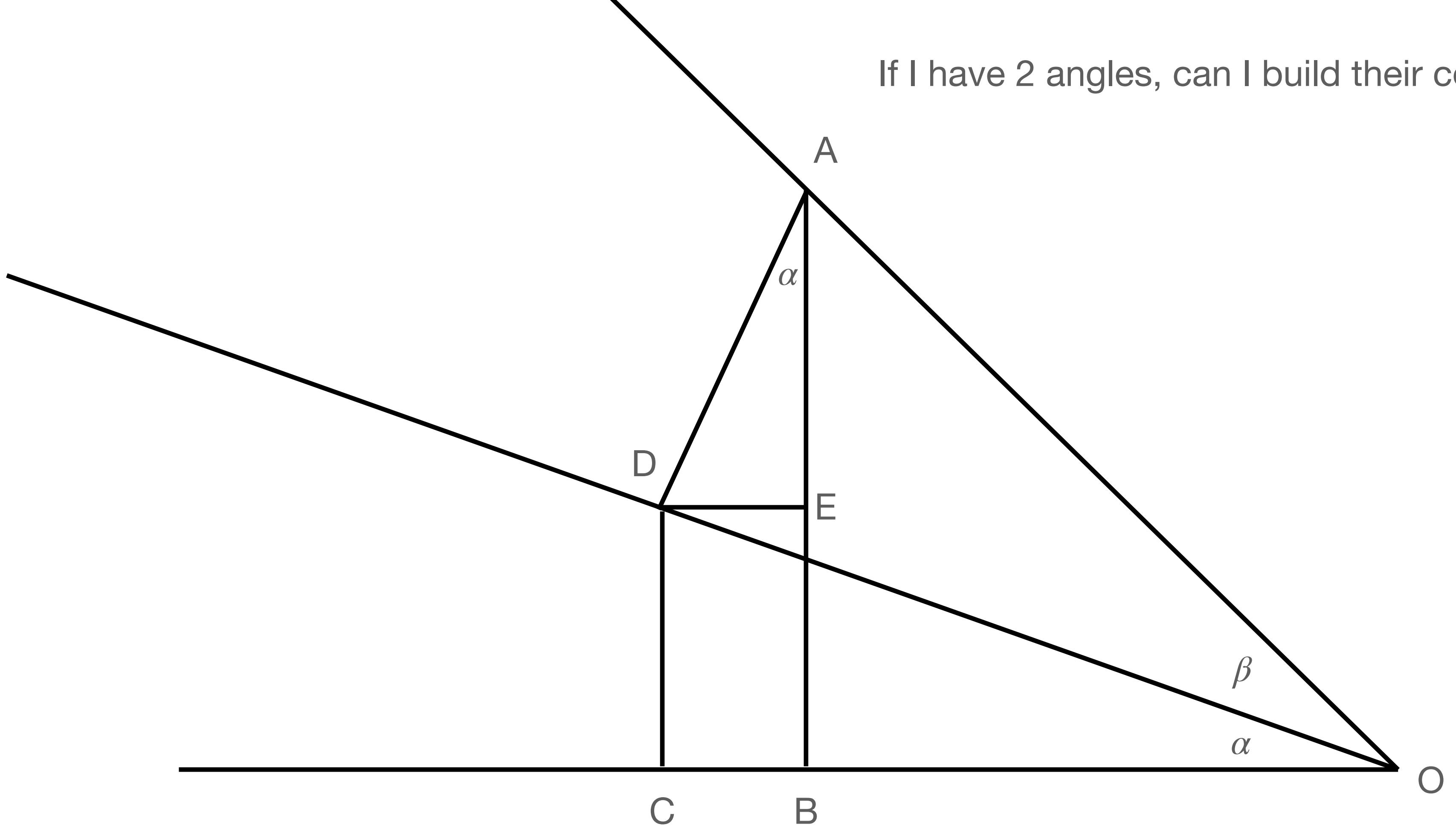

$$sin(\alpha + \beta) = \frac{AB}{AO}$$

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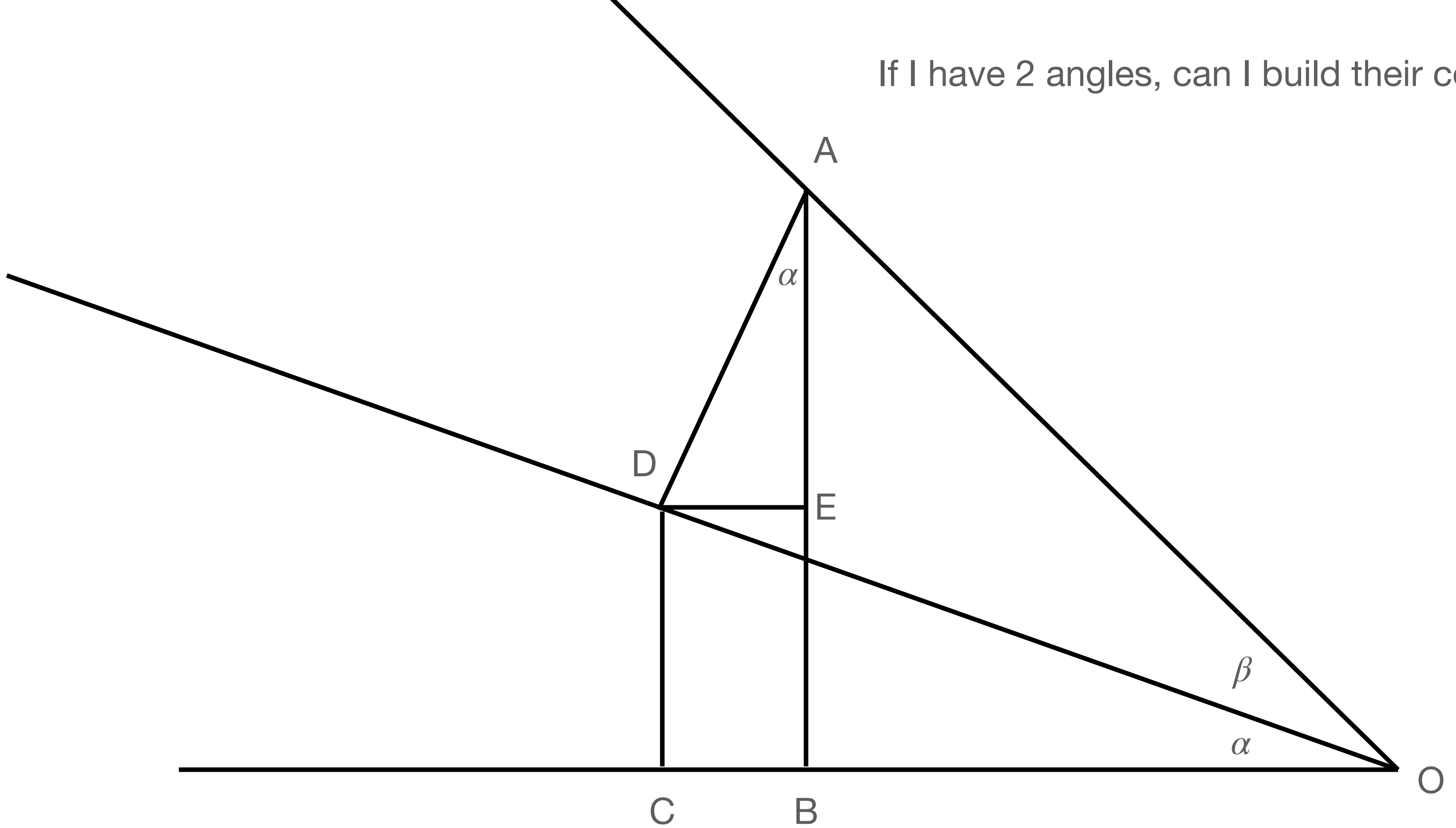

$$sin(\alpha + \beta) = \frac{AB}{AO}$$

$$= \frac{AE + EB}{AO}$$

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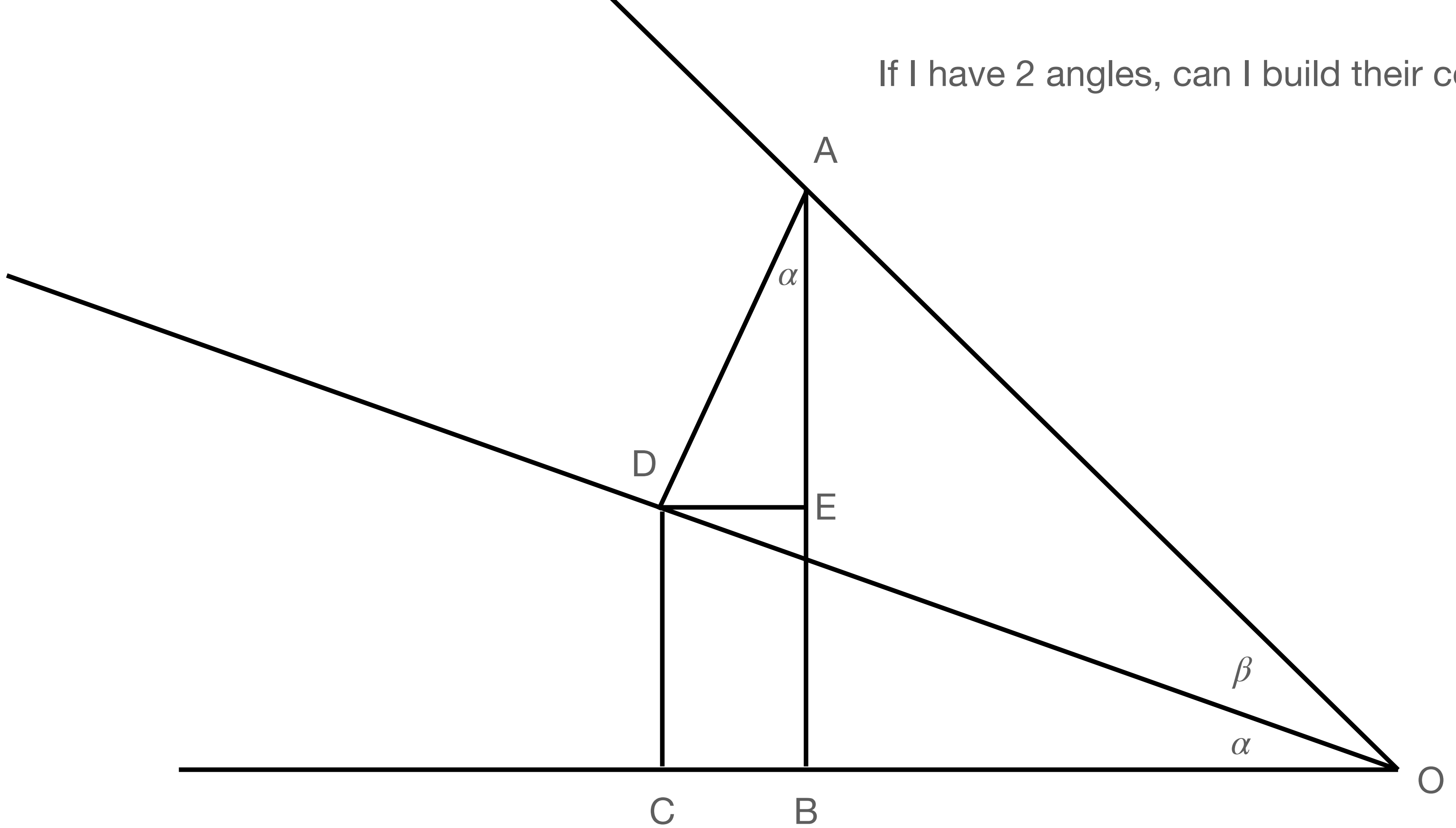

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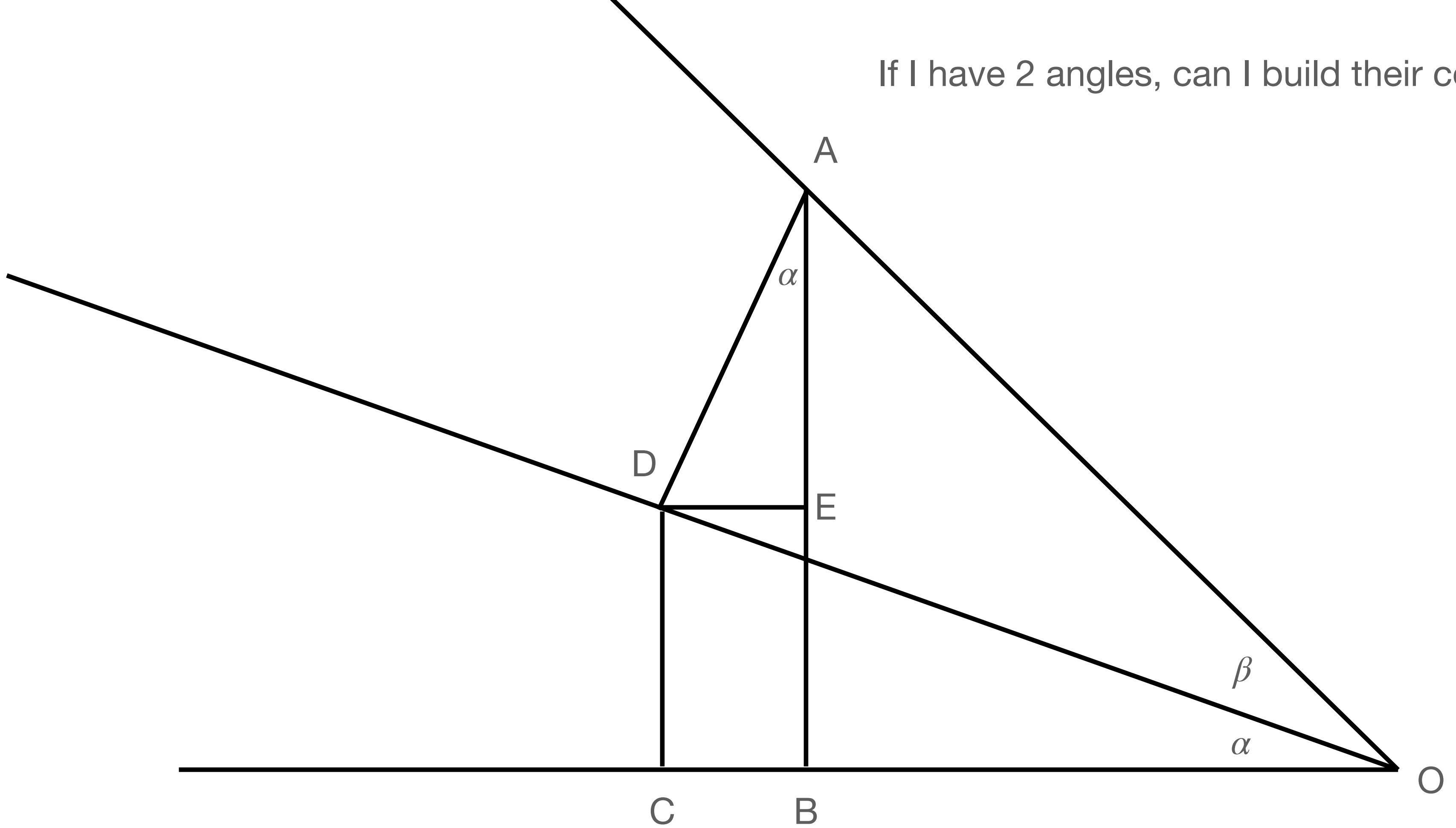

$$sin(\alpha+\beta) = \frac{AB}{AO}$$

$$= \frac{AE+EB}{AO}$$

$$= \frac{AE}{AO} + \frac{EB}{AO}$$

$$= \frac{AE}{AD} * \frac{AD}{AO} + \frac{EB}{AO}$$

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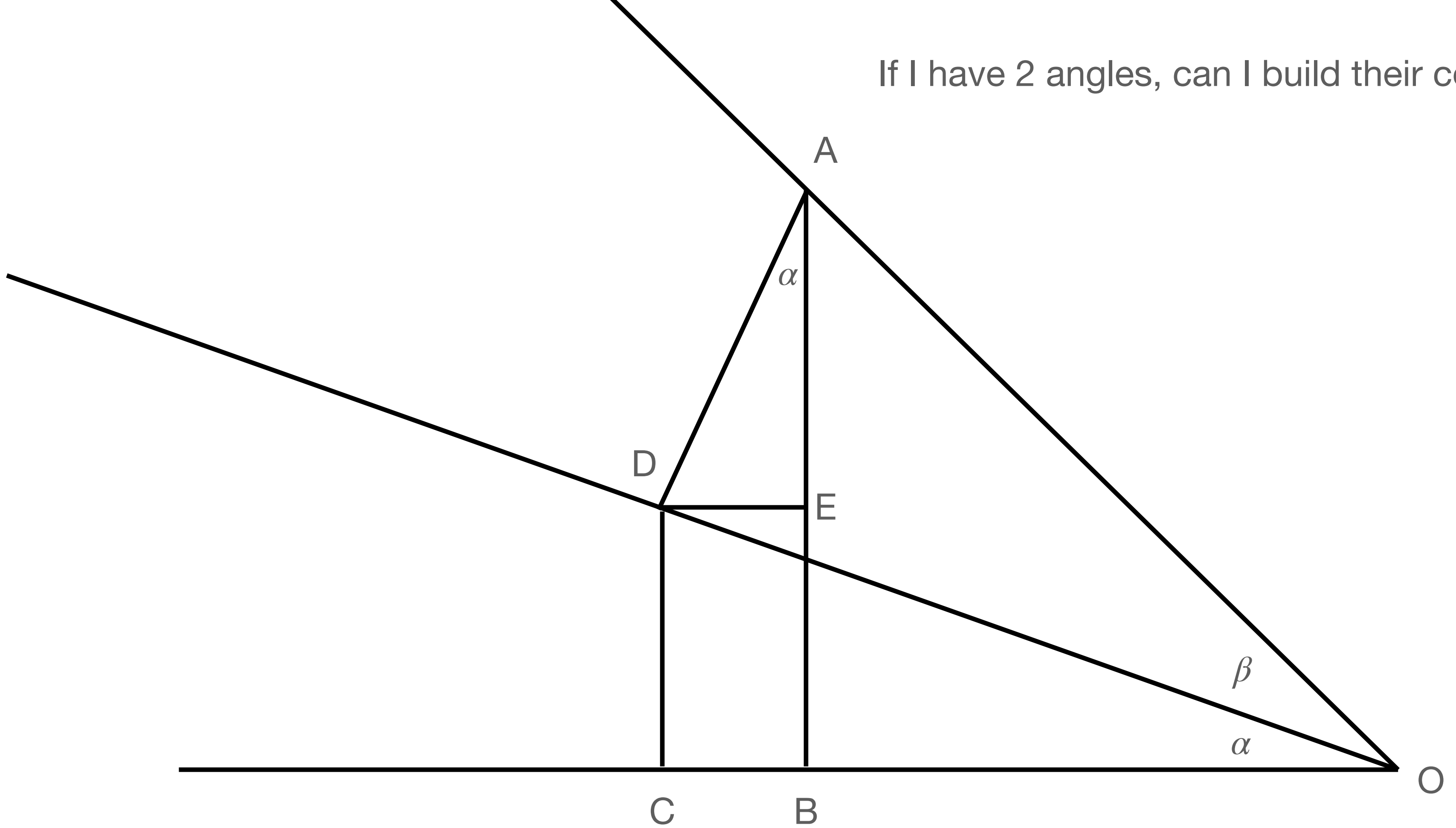

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$$= \frac{AE+EB}{AO}$$

$$= \frac{AE}{AO} + \frac{EB}{AO}$$

$$= \frac{AE}{AD} * \frac{AD}{AO} + \frac{EB}{AO}$$

$$= \frac{AE}{AD} * \frac{AD}{AO} + \frac{DC}{AO}$$

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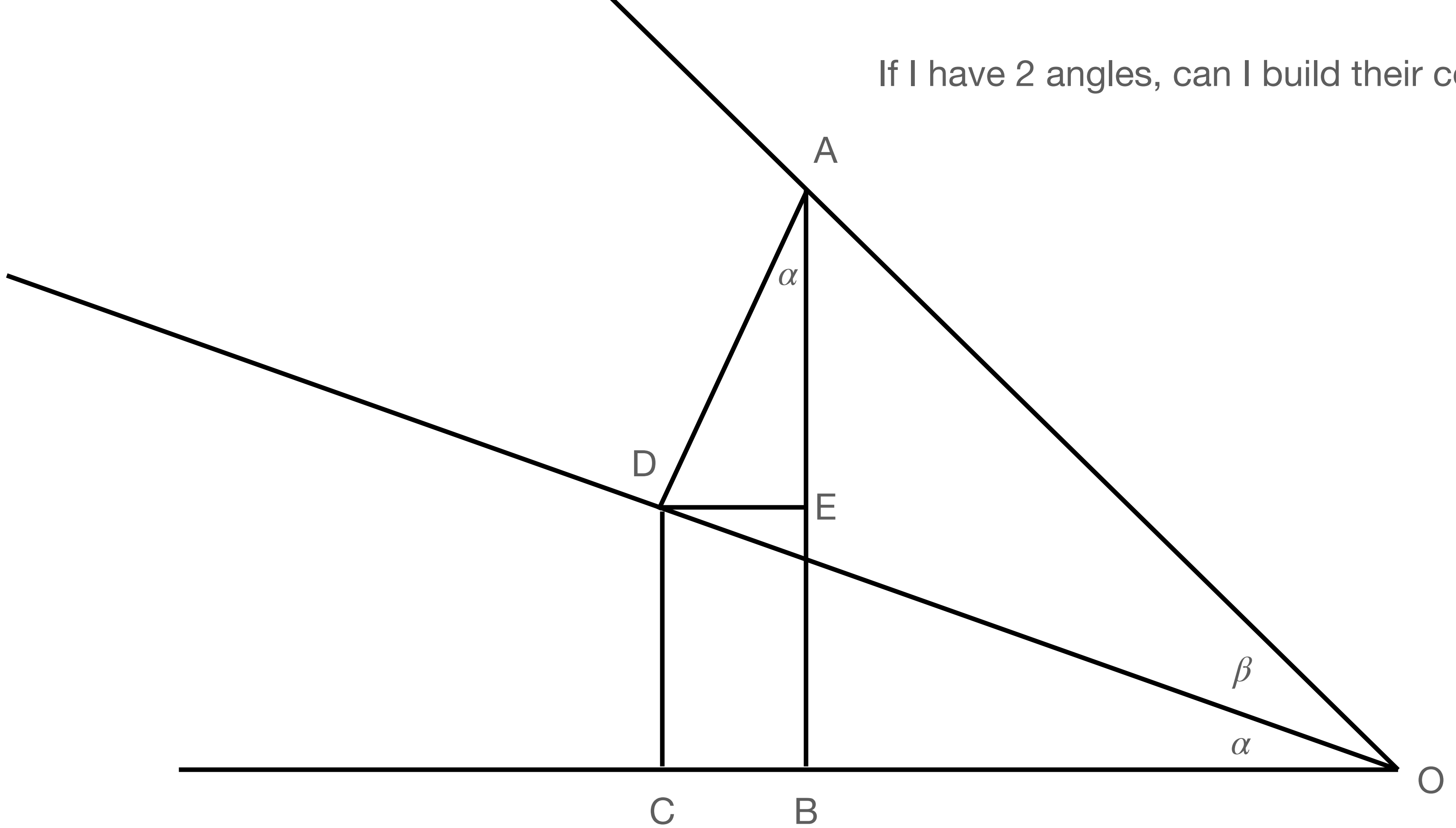

$$\begin{aligned}
sin(\alpha+\beta) &= \frac{AB}{AO} \\
&= \frac{AE+EB}{AO} \\
&= \frac{AE}{AO}+\frac{EB}{AO} \\
&= \frac{AE}{AD}*\frac{AD}{AO}+\frac{EB}{AO} \\
&= \frac{AE}{AD}*\frac{AD}{AO}+\frac{DC}{AO} \\
&= \frac{AE}{AD}*\frac{AD}{AO} + \frac{DC}{DO}*\frac{DO}{AO}
\end{aligned}$$

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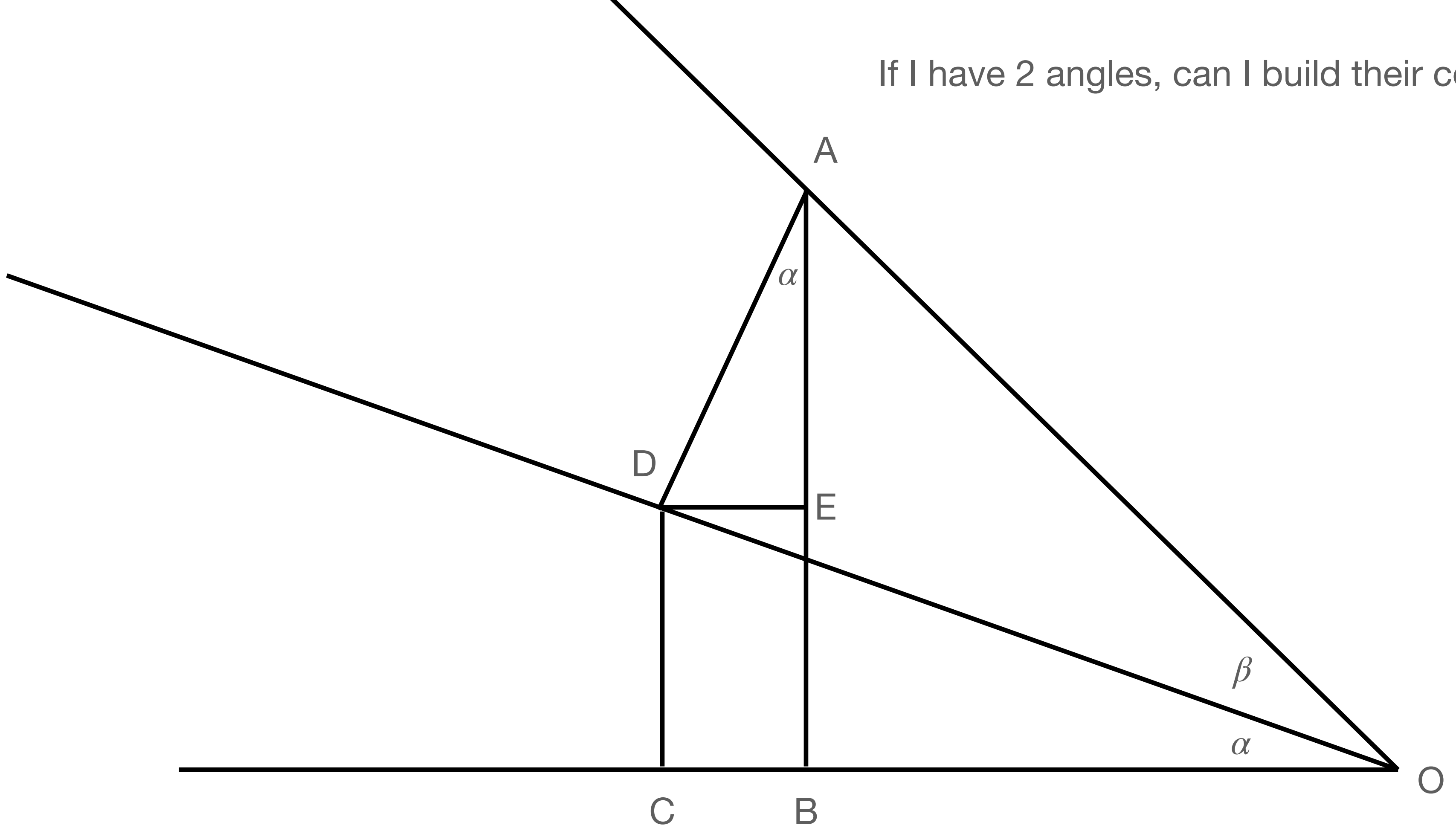


$$\sin(\alpha + \beta) = \frac{AB}{AO}$$

$$= \frac{AE + EB}{AO}$$

$$= \frac{AE}{AO} + \frac{EB}{AO}$$

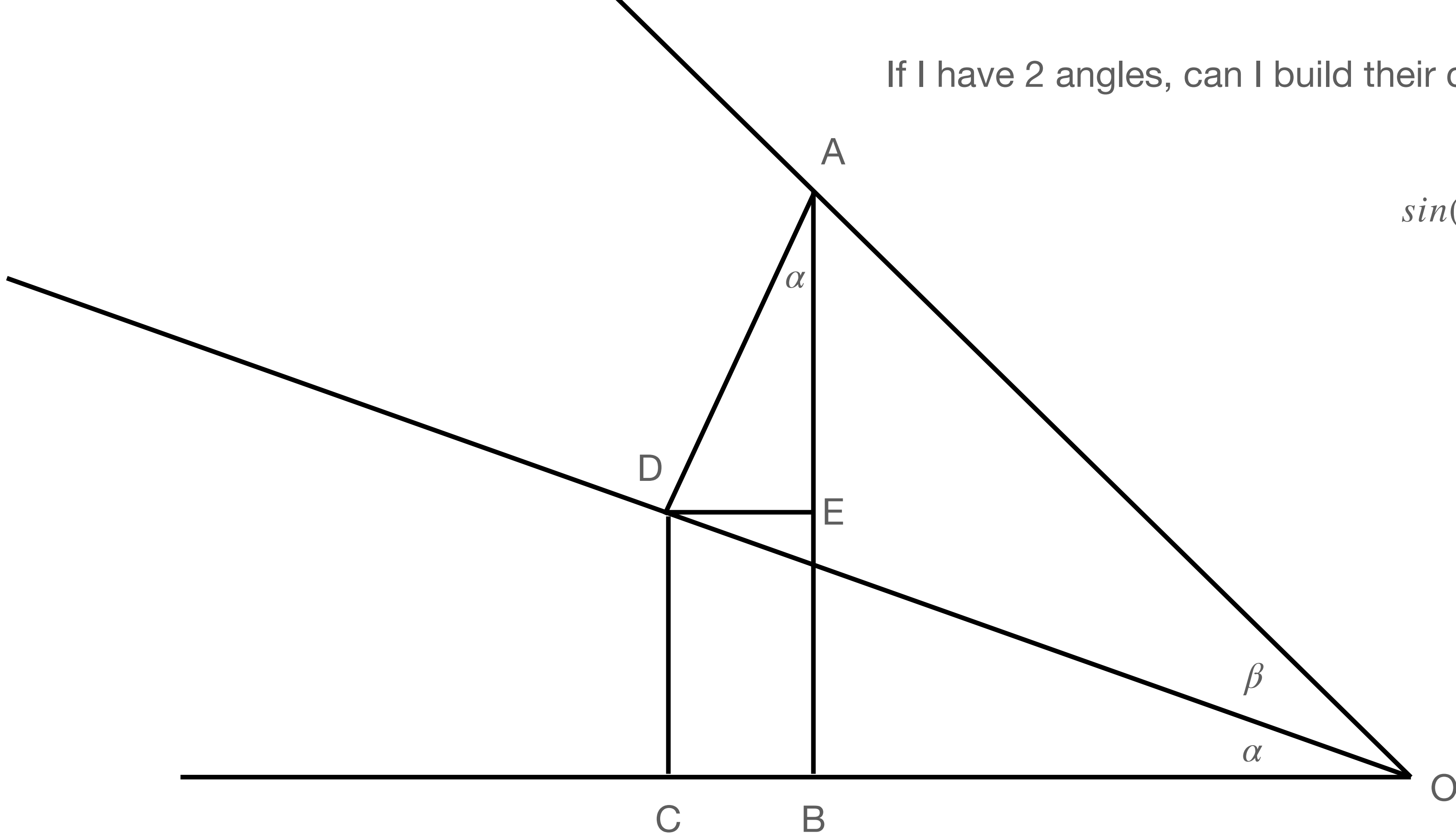
$$= \frac{AE}{AD} * \frac{AD}{AO} + \frac{EB}{AO}$$

$$= \frac{AE}{AD} * \frac{AD}{AO} + \frac{DC}{AO}$$

$$= \frac{AE}{AD} * \frac{AD}{AO} + \frac{DC}{DO} * \frac{DO}{AO} \qquad = \cos(\alpha) * \sin(\beta) + \sin(\alpha) * \cos(\beta)$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha + \beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$



$$sin(\alpha + \beta) = \frac{AB}{AO}$$

$$= \frac{AE + EB}{AO}$$

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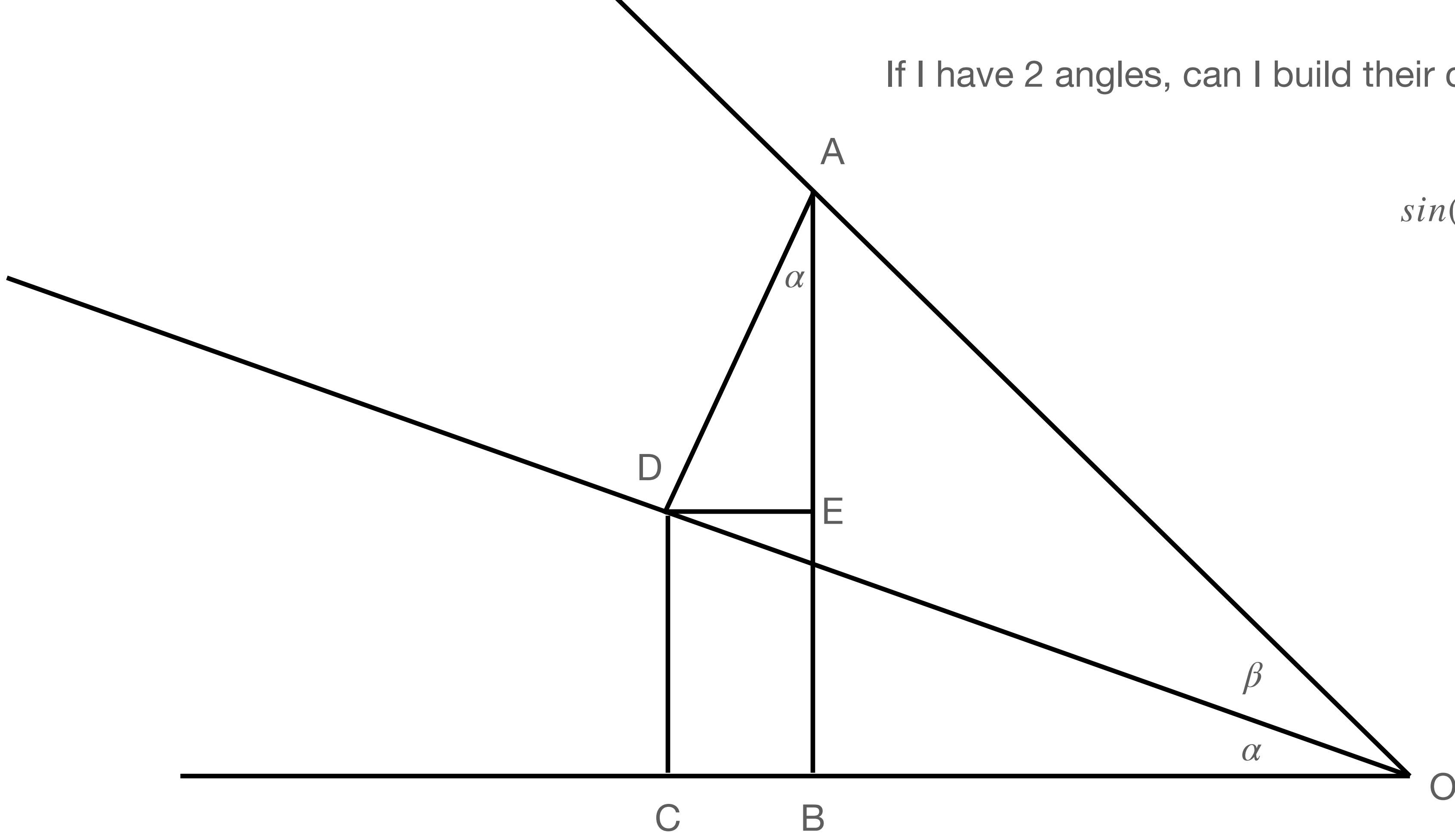

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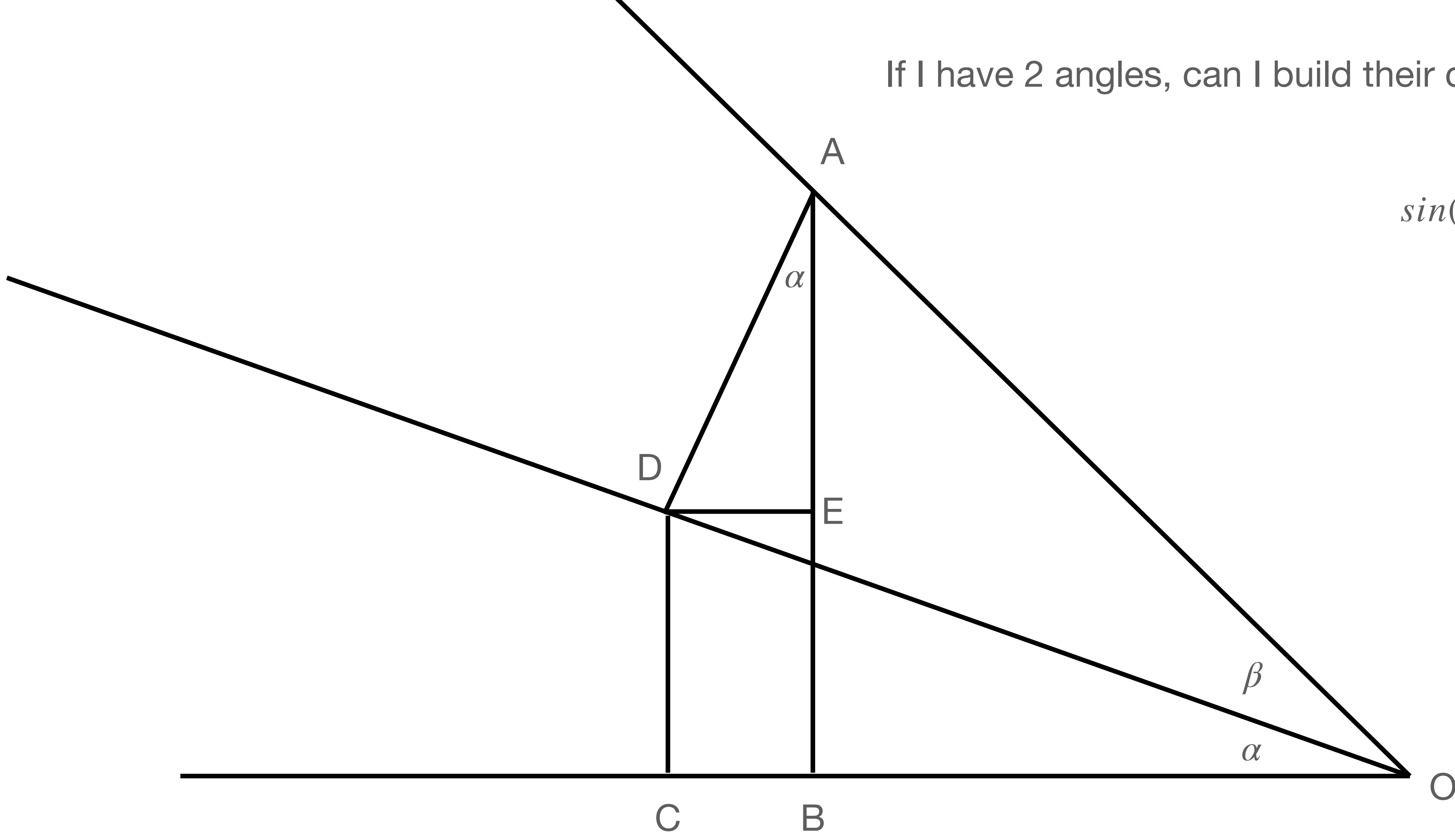
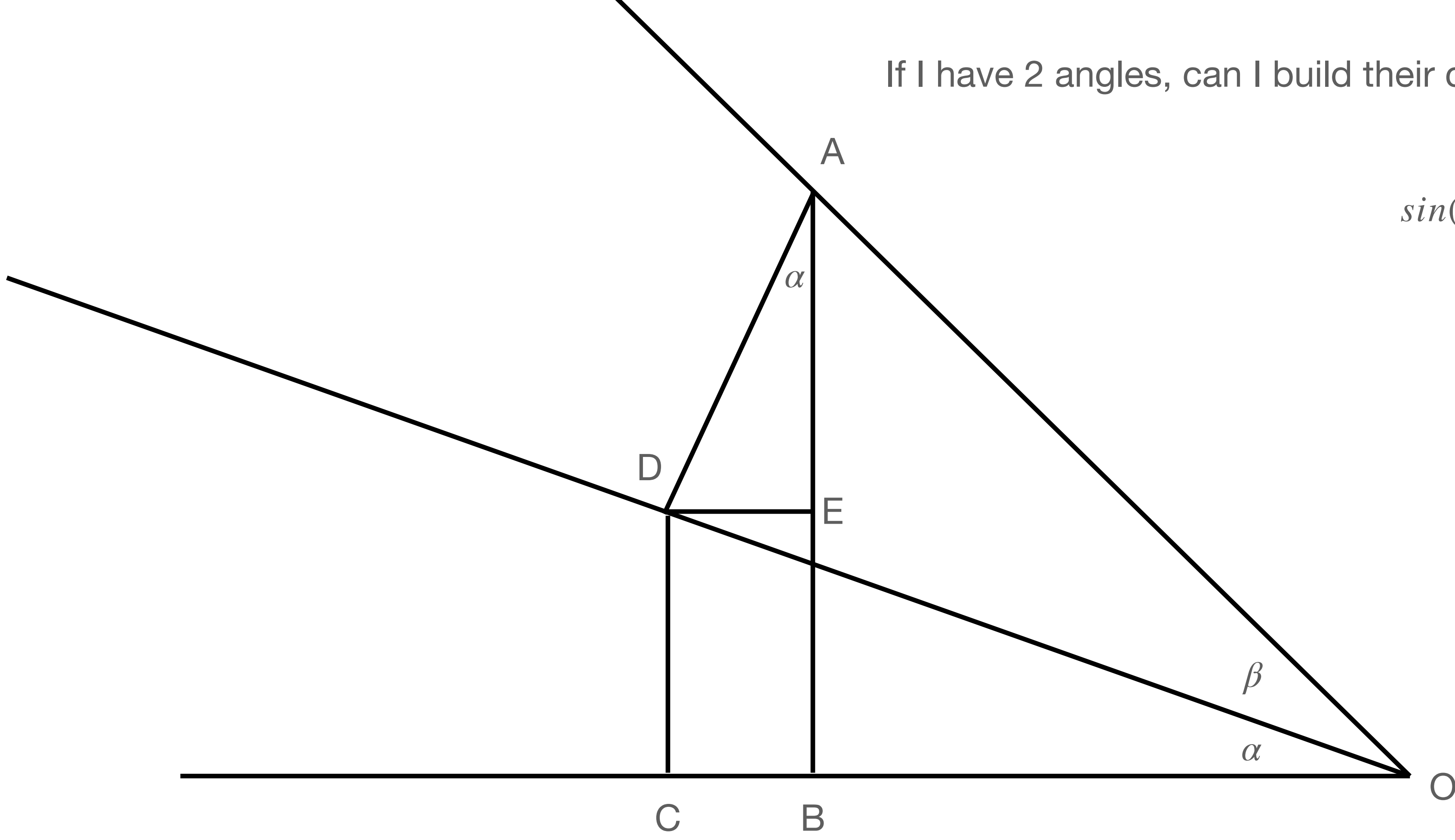
$$sin(\alpha+\beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$



$$cos(\alpha+\beta) = \frac{OB}{AO}$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha + \beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$

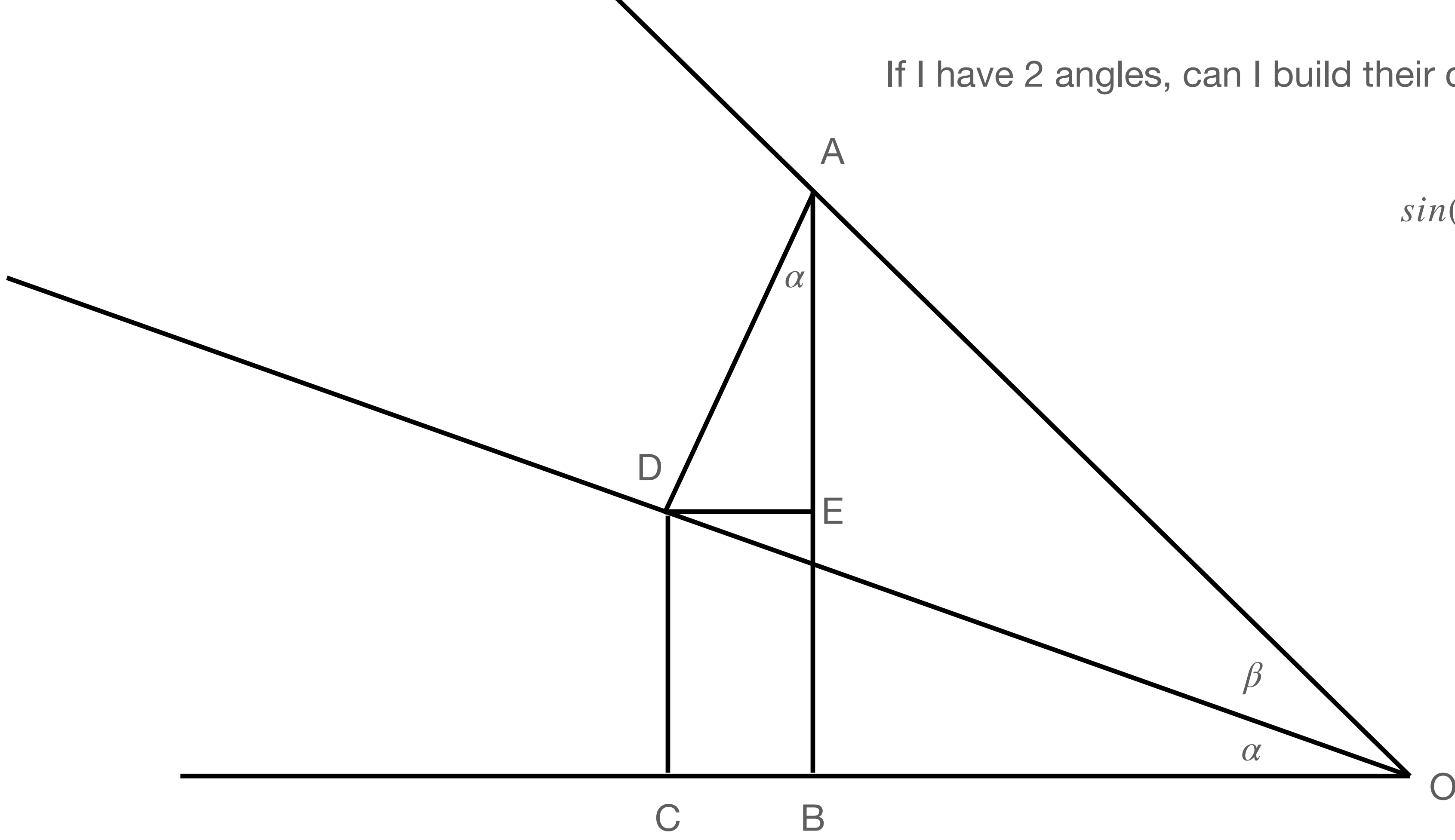


$$cos(\alpha + \beta) = \frac{OB}{AO}$$

$$= \frac{OC - CB}{AO}$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha + \beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$



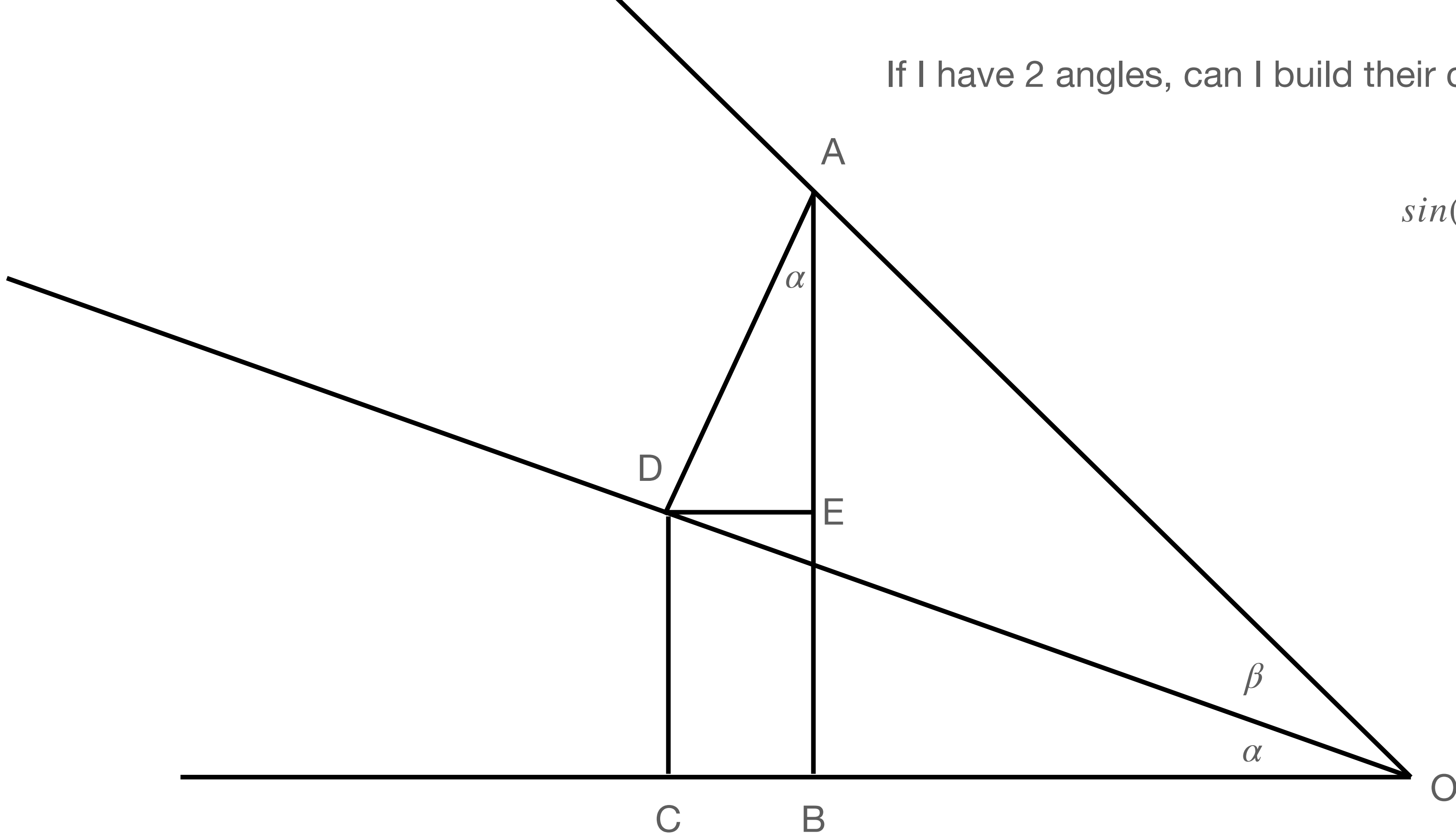
$$cos(\alpha + \beta) = \frac{OB}{AO}$$

$$= \frac{OC - CB}{AO}$$

$$= \frac{OC}{AO} - \frac{CB}{AO}$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha+\beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$



$$cos(\alpha+\beta) = \frac{OB}{AO}$$

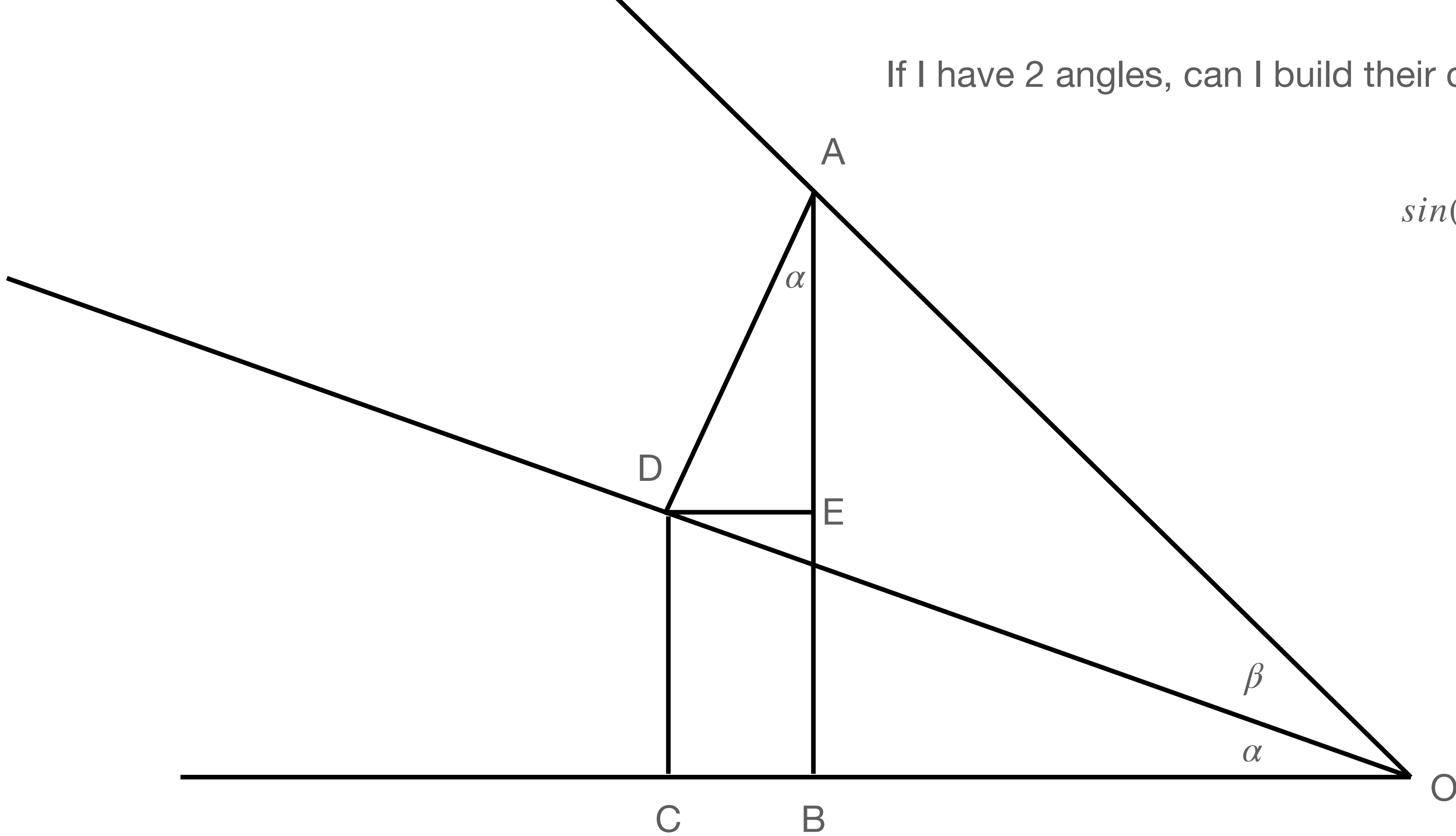
$$= \frac{OC - CB}{AO}$$

$$= \frac{OC}{AO} - \frac{CB}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AO}$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha+\beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$



$$cos(\alpha+\beta) = \frac{OB}{AO}$$

$$= \frac{OC - CB}{AO}$$

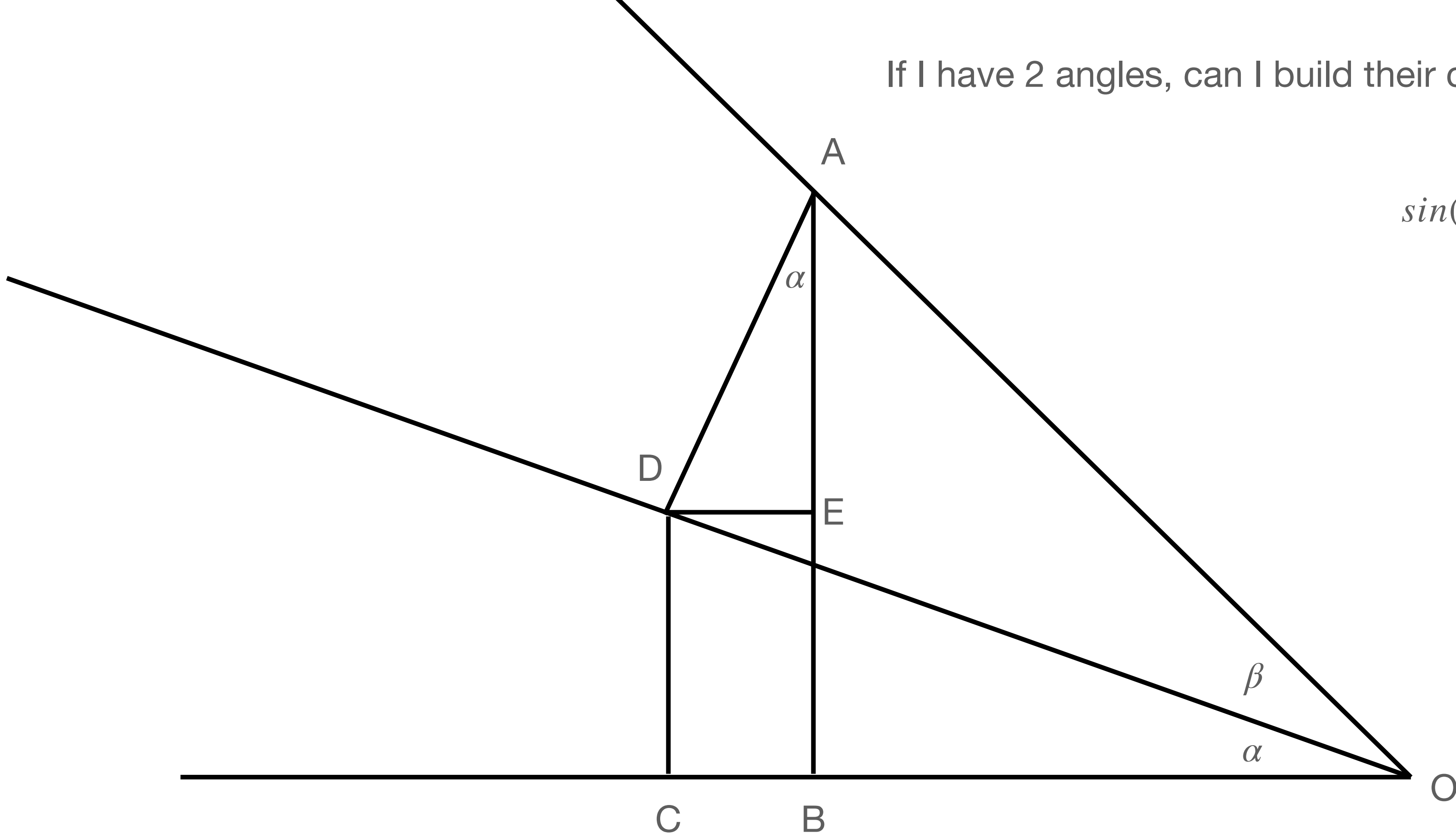
$$= \frac{OC}{AO} - \frac{CB}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AD} * \frac{AD}{AO}$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha+\beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$



$$cos(\alpha+\beta) = \frac{OB}{AO}$$

$$= \frac{OC - CB}{AO}$$

$$= \frac{OC}{AO} - \frac{CB}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AD} * \frac{AD}{AO}$$

$$= cos(\alpha) * cos(\beta) - sin(\alpha) * sin(\beta)$$

If I have 2 angles, can I build their combined properties from the single angles?

$$sin(\alpha + \beta) = cos(\alpha) * sin(\beta) + sin(\alpha) * cos(\beta)$$

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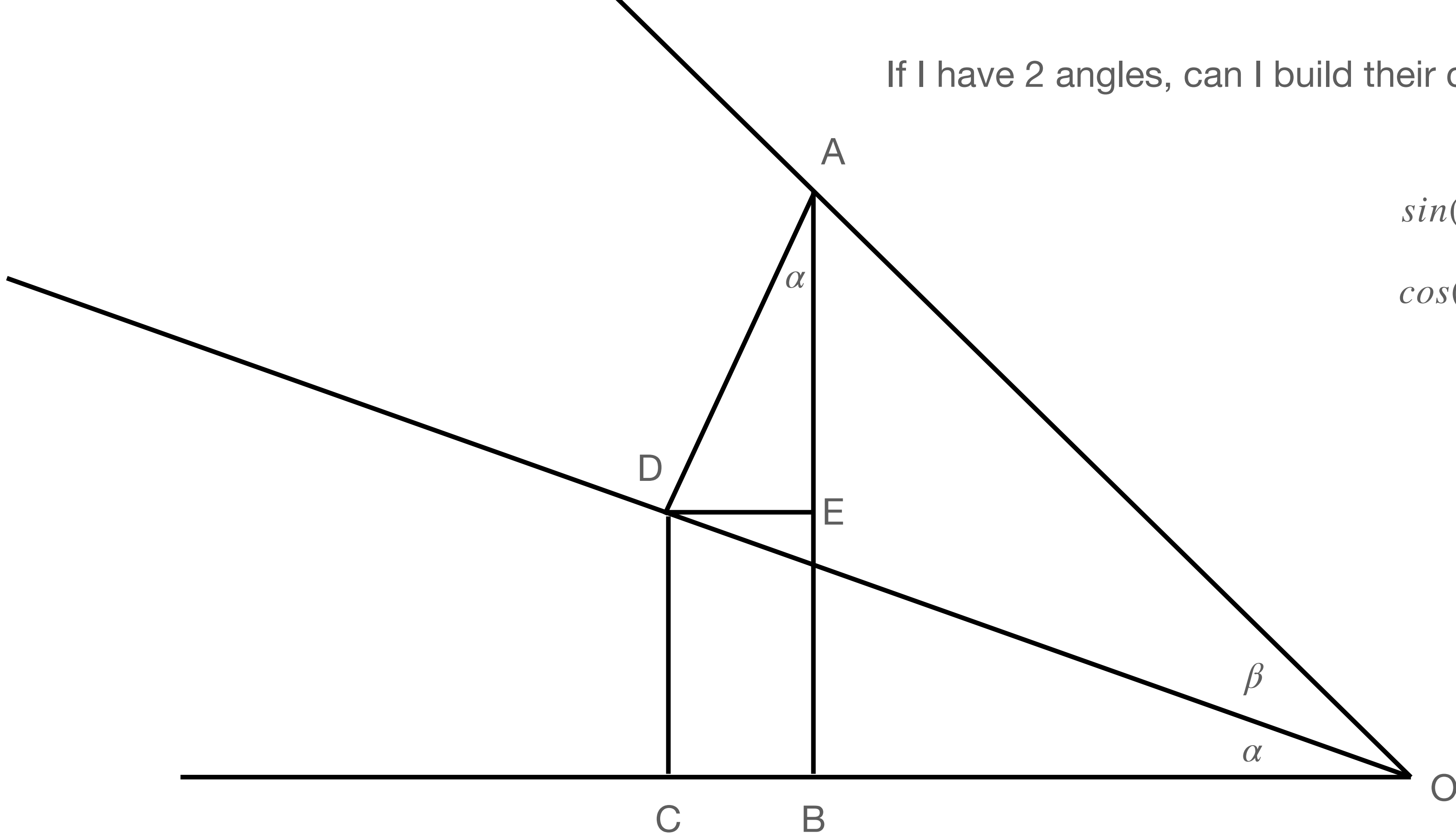

$$cos(\alpha + \beta) = \frac{OB}{AO}$$

$$= \frac{OC - CB}{AO}$$

$$= \frac{OC}{AO} - \frac{CB}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AO}$$

$$= \frac{OC}{OD} * \frac{OD}{AO} - \frac{DE}{AD} * \frac{AD}{AO}$$

$$= cos(\alpha) * cos(\beta) - sin(\alpha) * sin(\beta)$$

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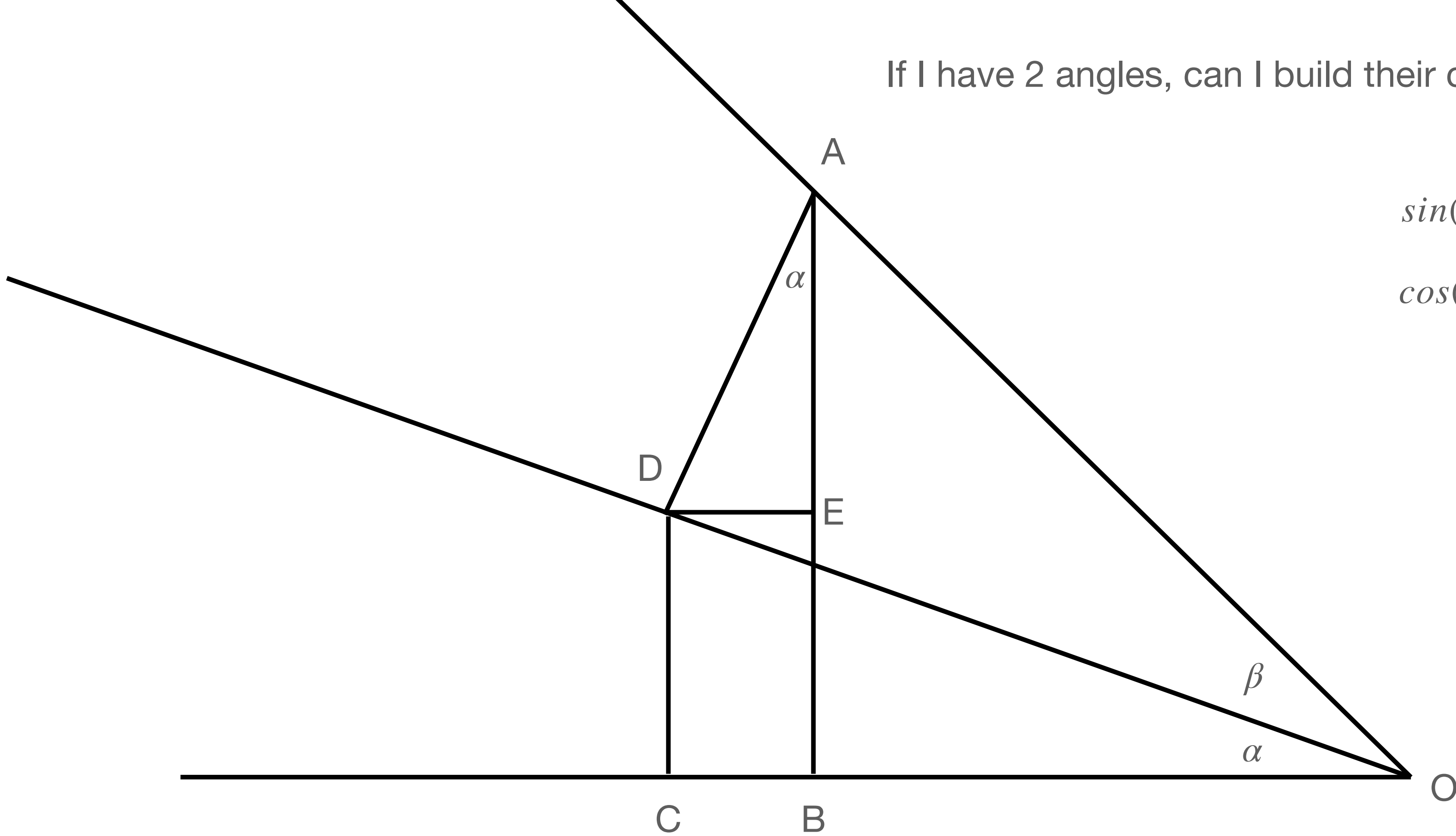

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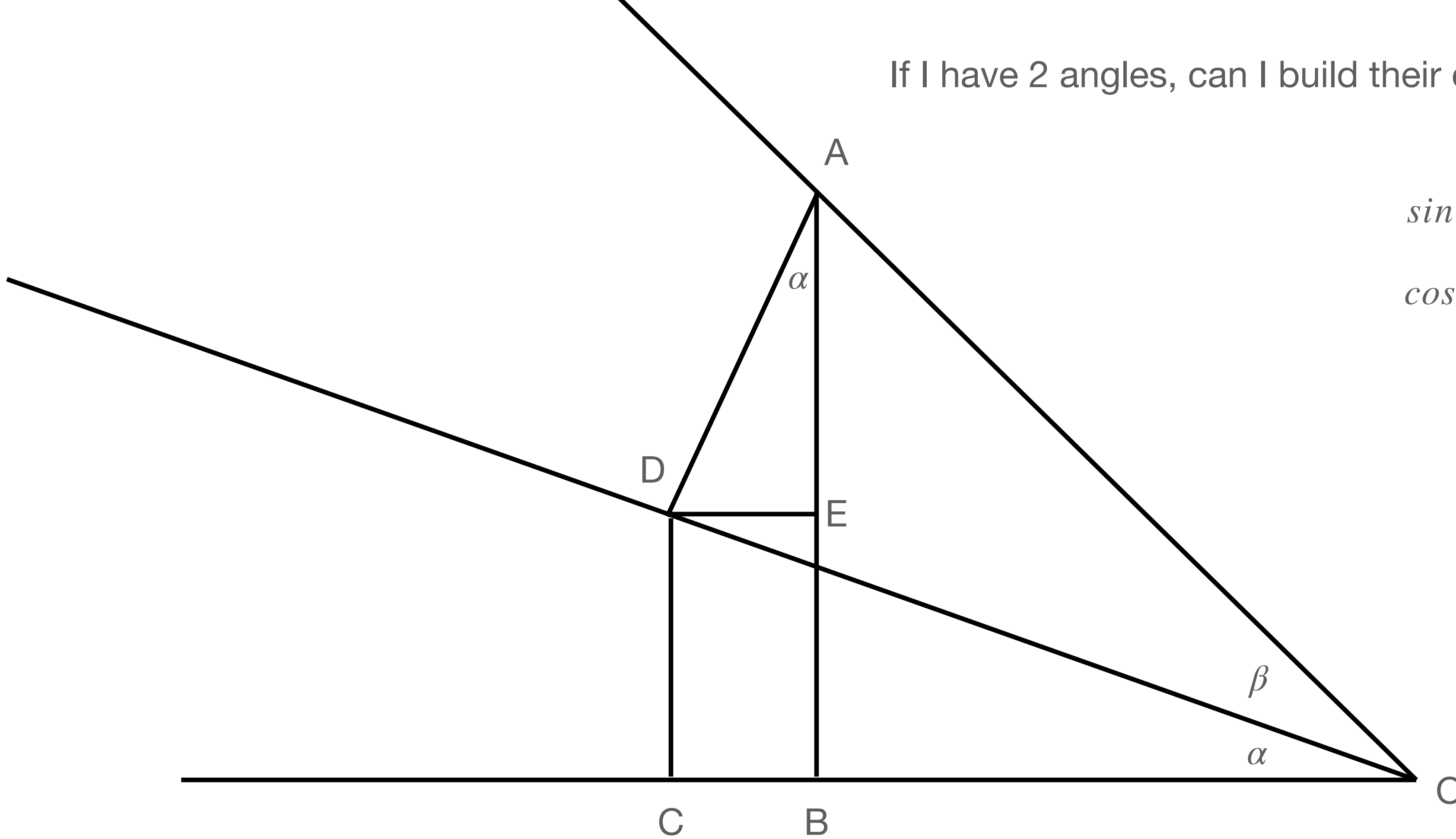

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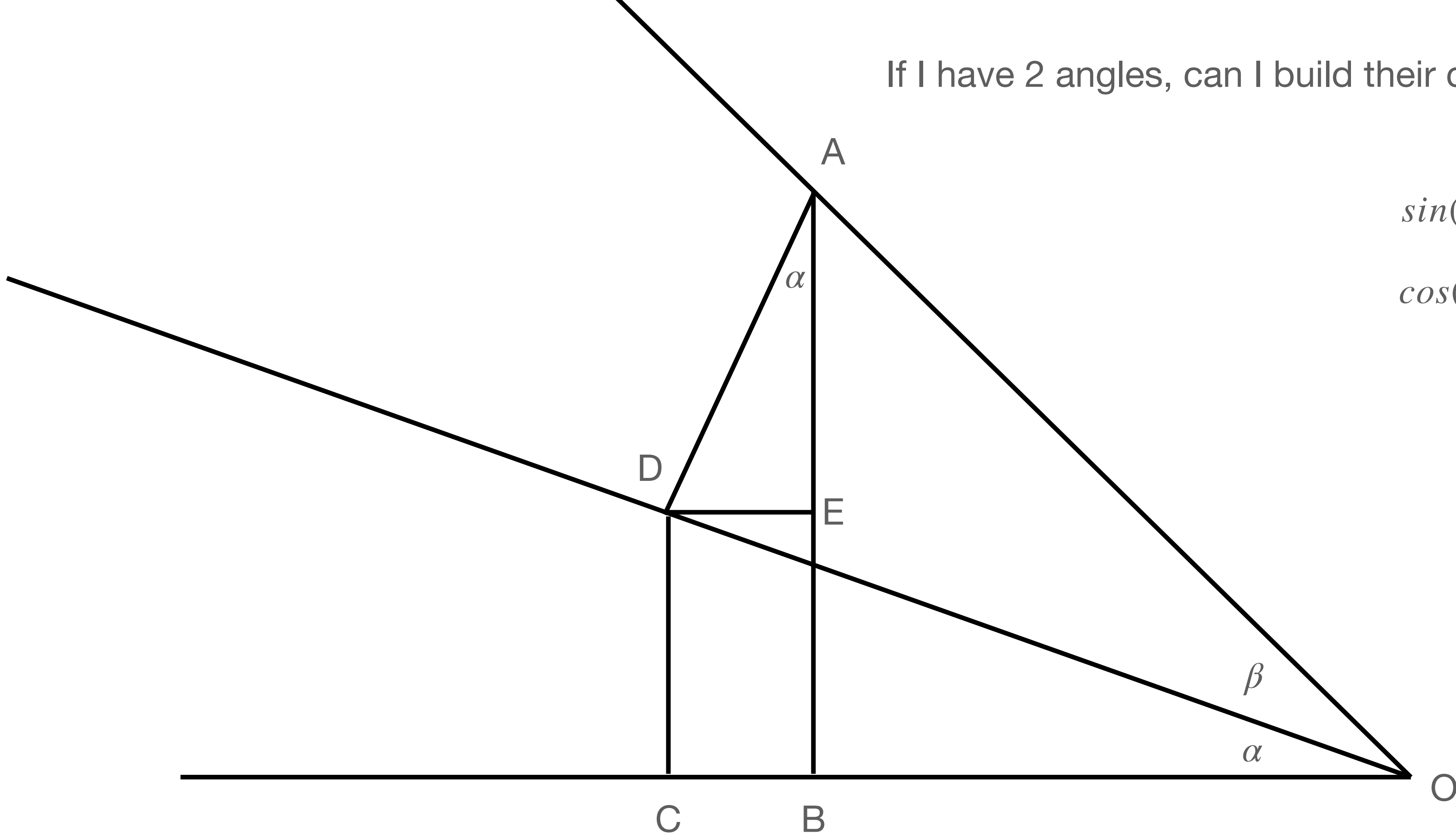

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$a = cos(\alpha)$ $b = sin(\alpha)$

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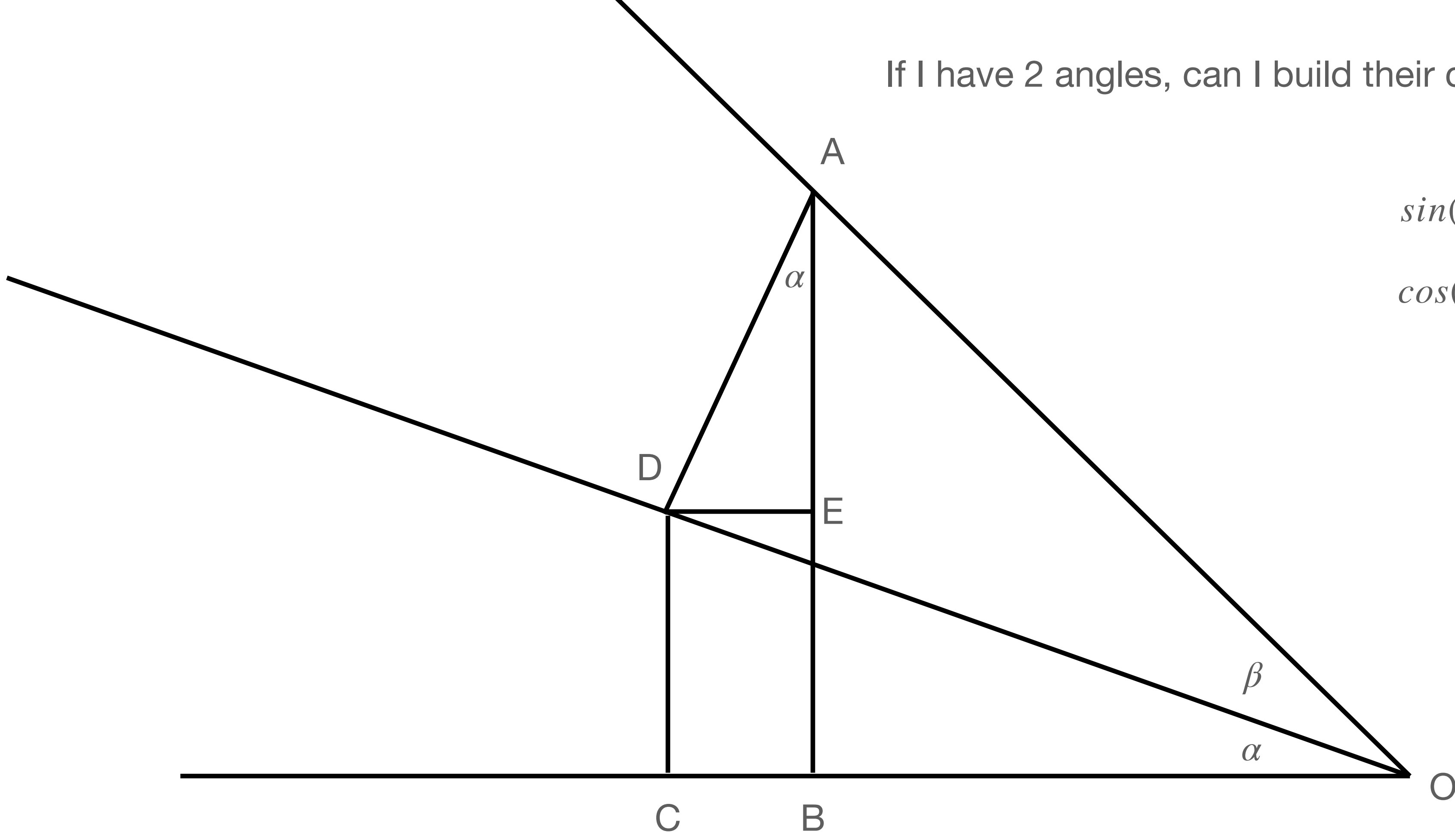

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$$sin(\alpha+\beta) = a * sin(\beta) + b * cos(\beta)$$

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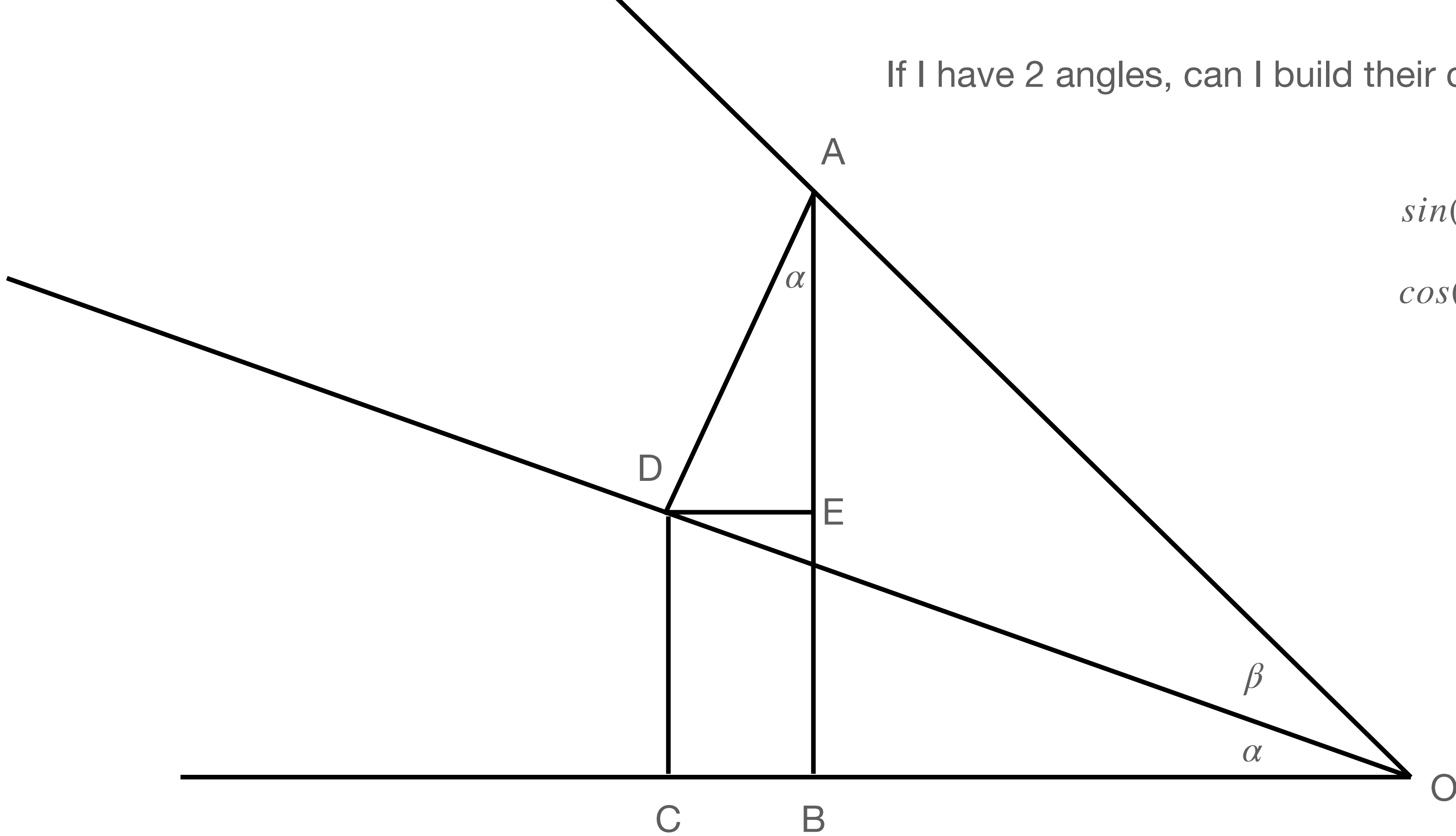

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$a = cos(\alpha)$ $\quad b = sin(\alpha)$

$g = cos(\beta)$ $\quad h = sin(\beta)$

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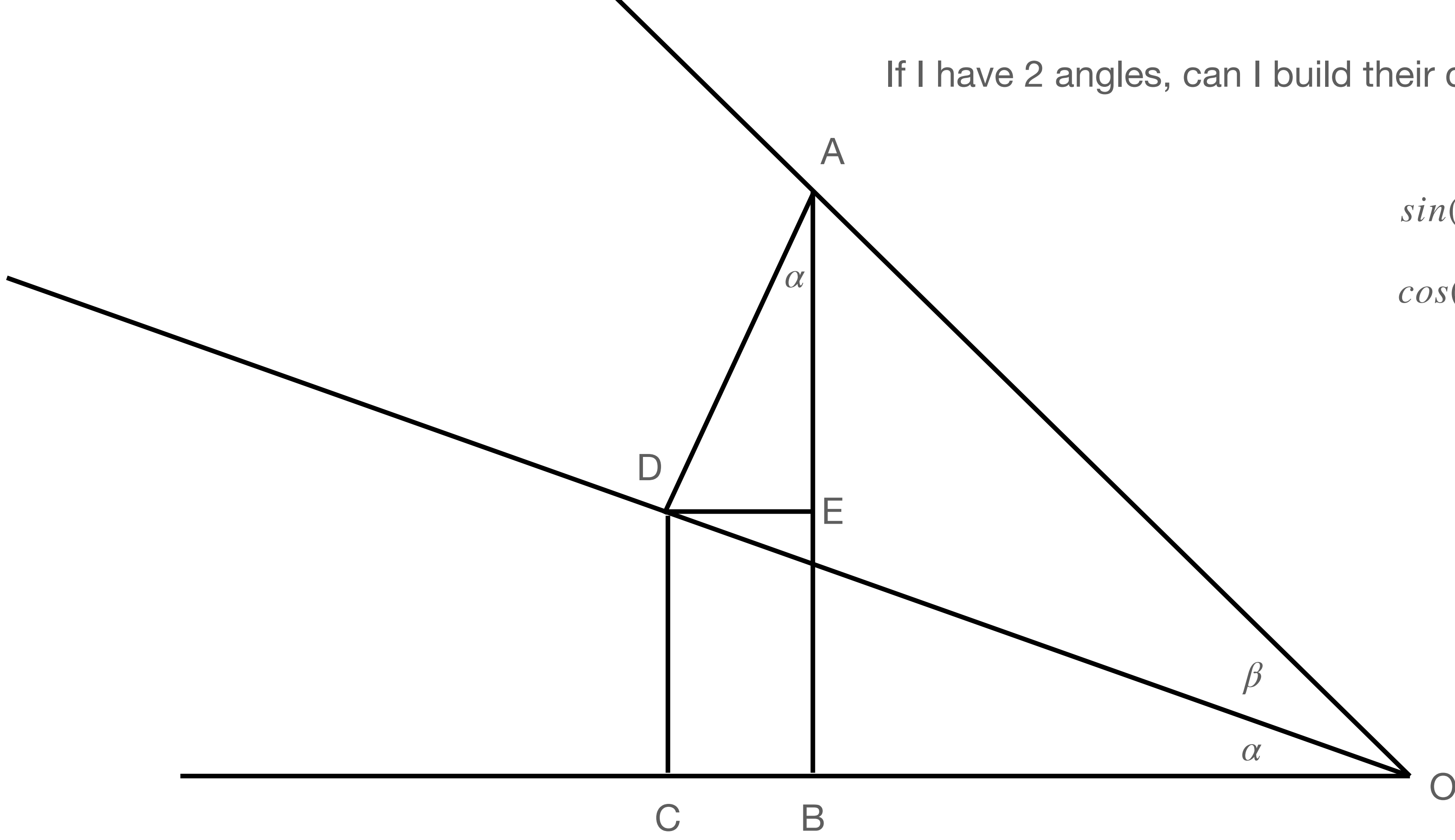

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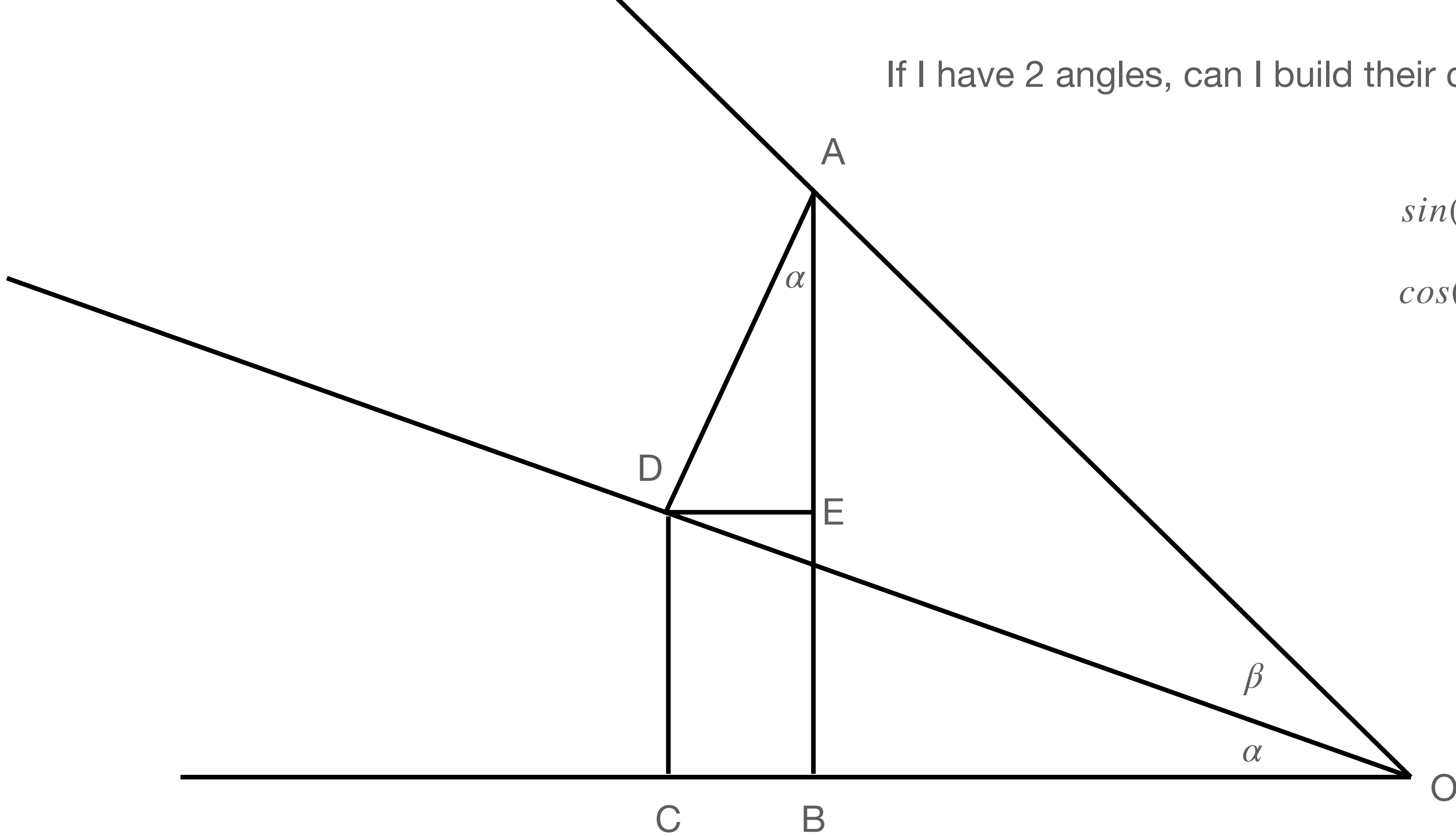

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$a = cos(\alpha)$ $b = sin(\alpha)$ $(a, b) = (cos(\alpha), sin(\alpha))$

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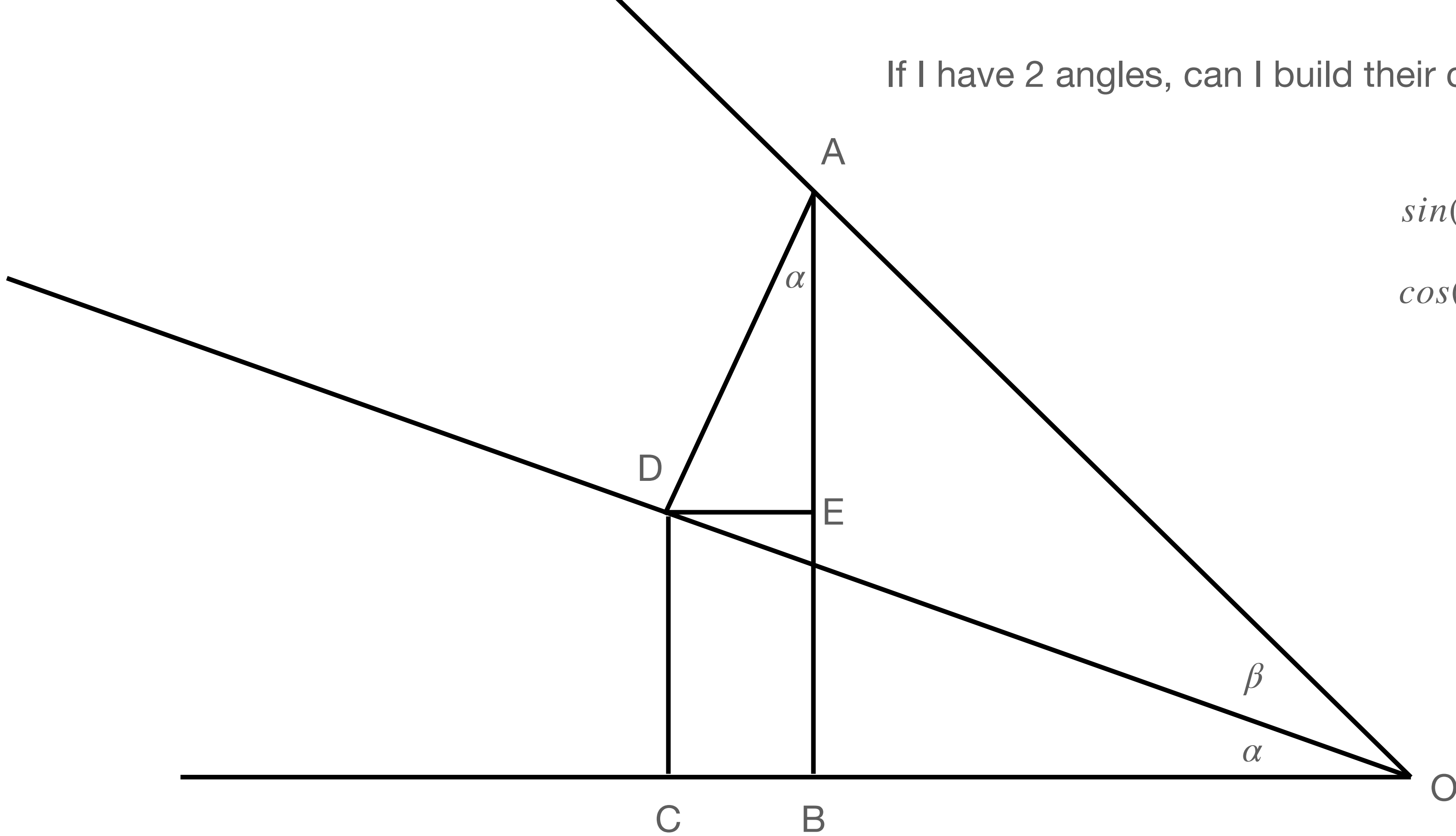

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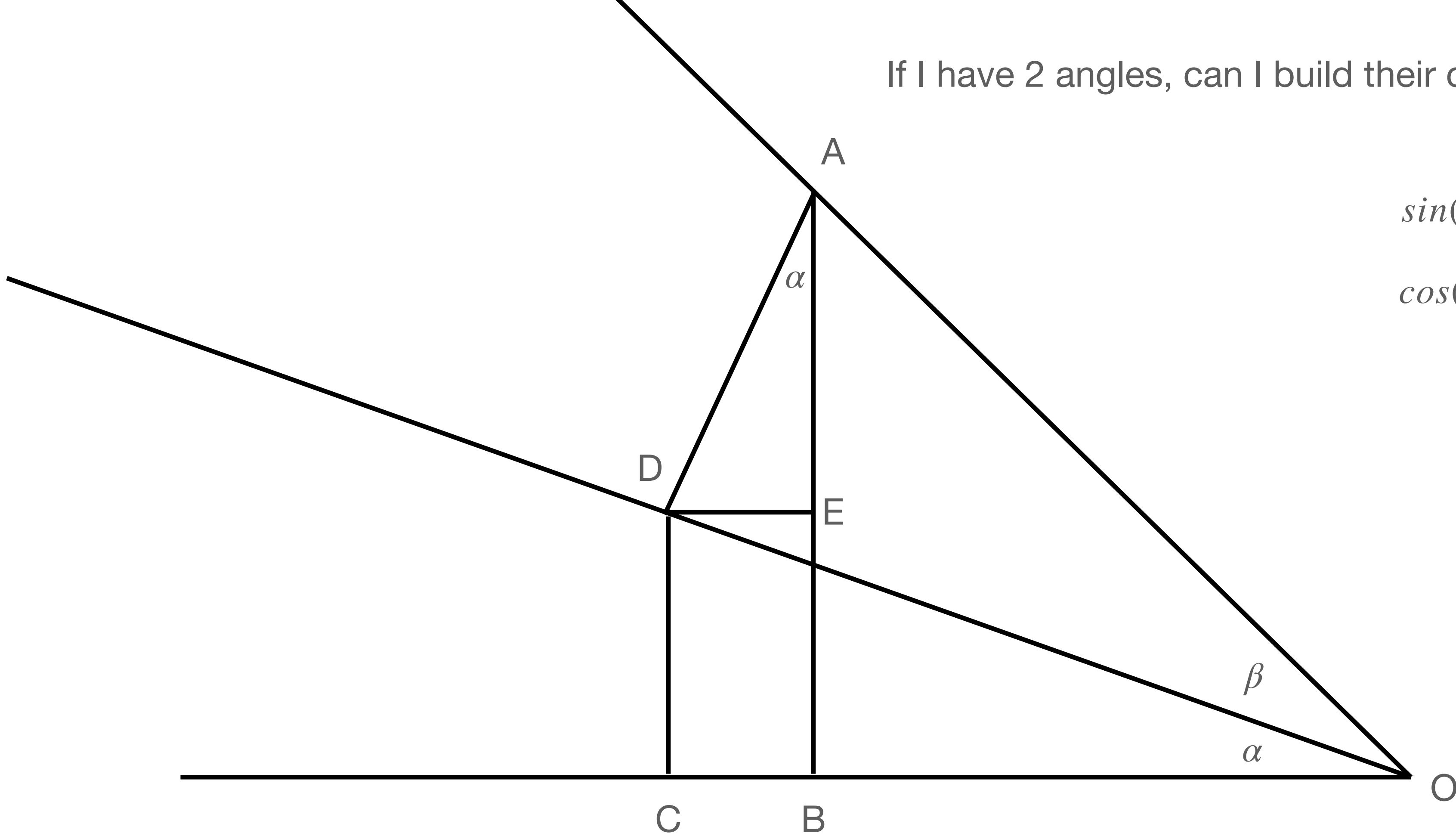

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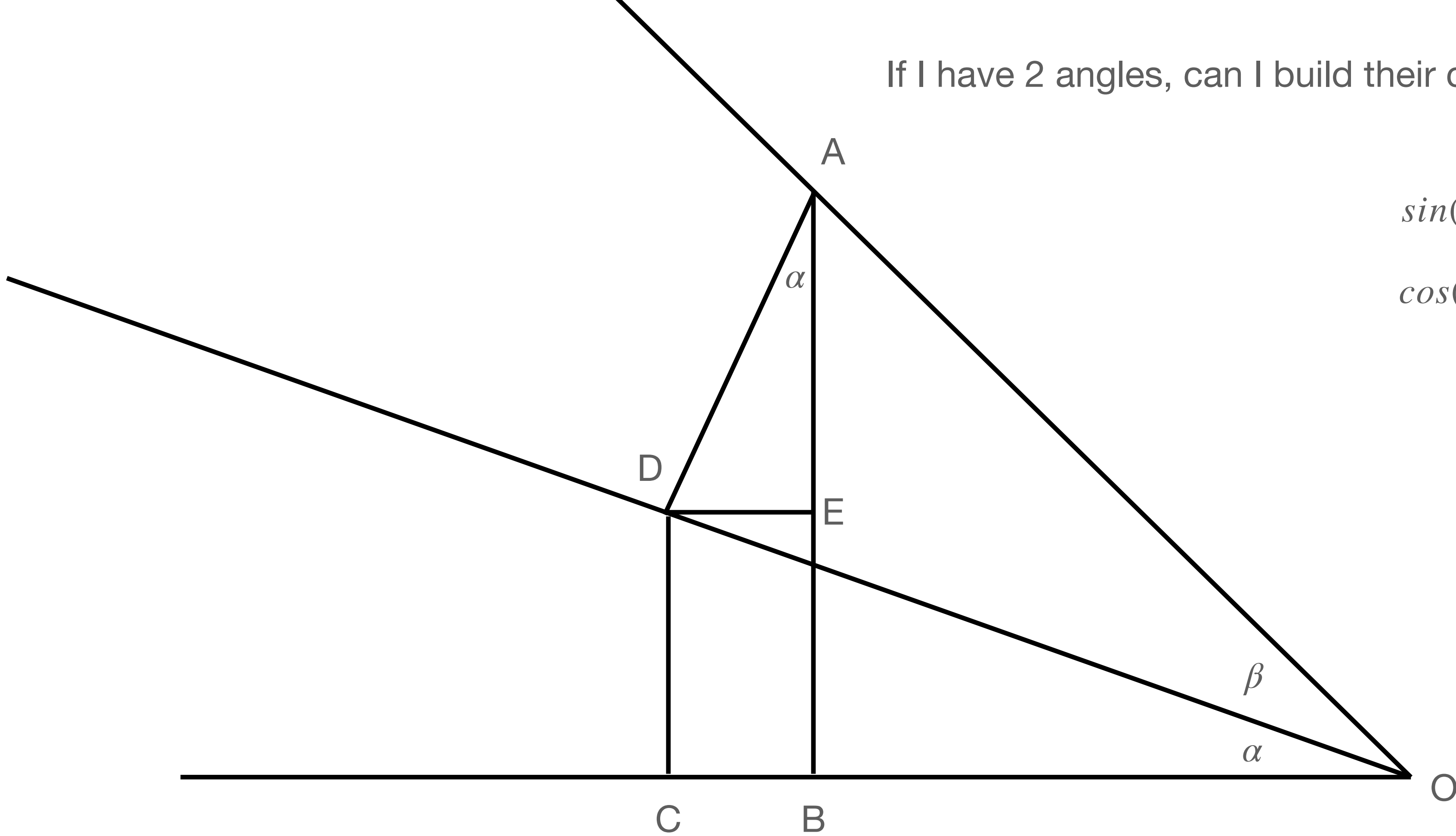
$$(cos(\alpha + \beta), sin(\alpha + \beta))$$

If I have 2 angles, can I build their combined properties from the single angles?



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$$(cos(\alpha + \beta), sin(\alpha + \beta)) = (ag - bh, ah + bg)$$

If I have 2 angles, can I build their combined properties from the single angles?

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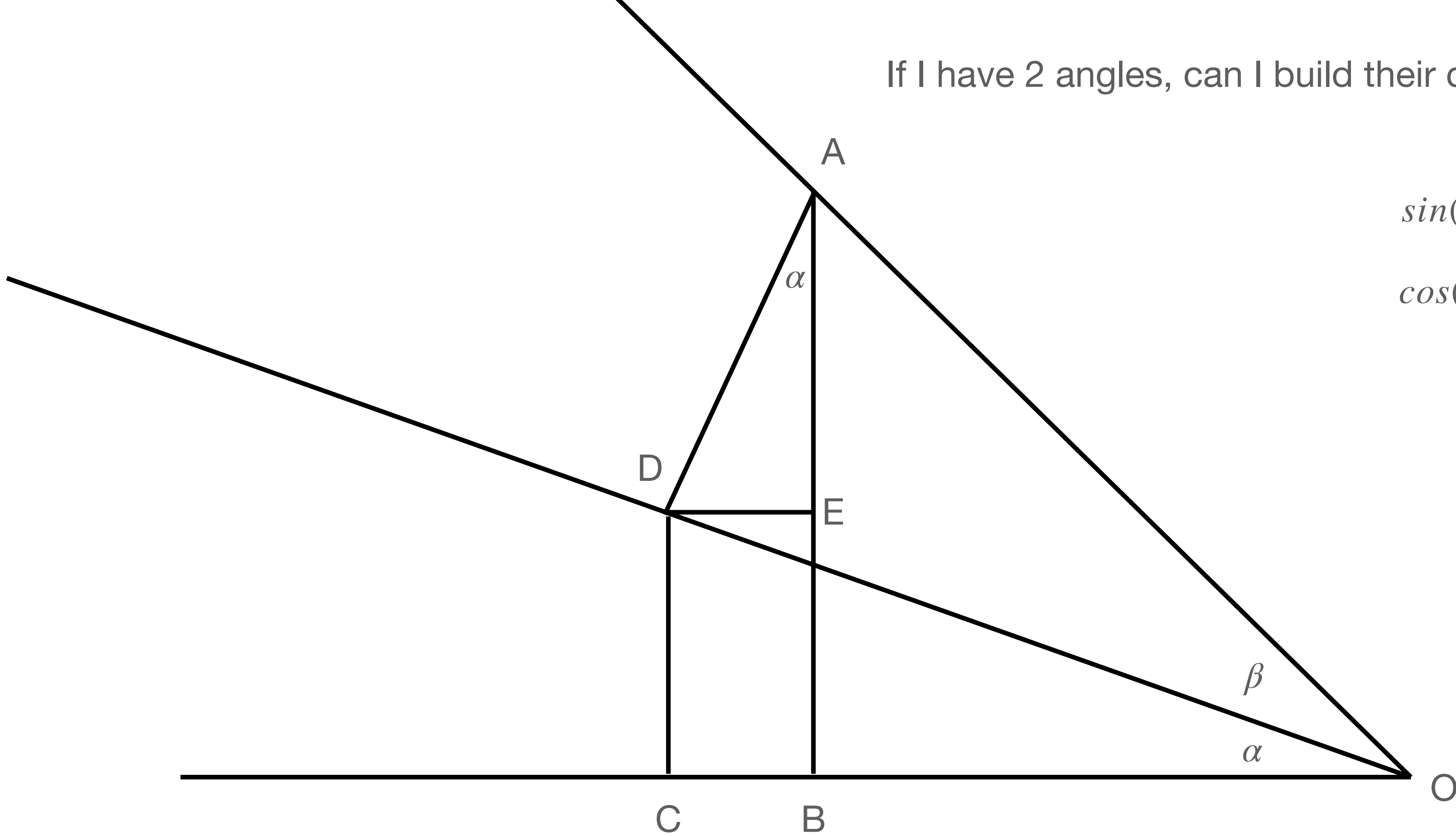

$$sin(\alpha+\beta) = a * h + b * g$$

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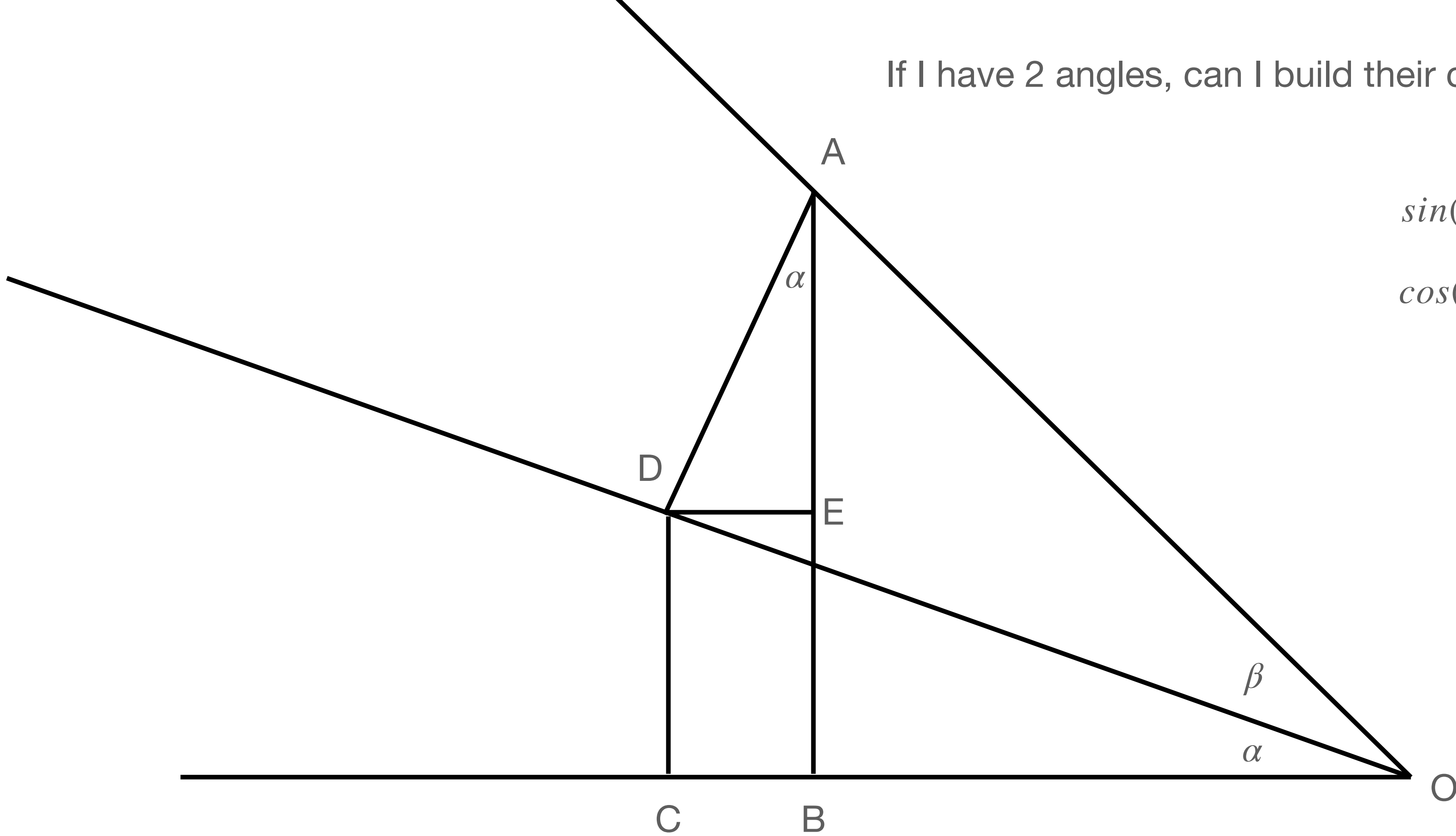
$$(cos(\alpha+\beta), sin(\alpha+\beta)) = (ag - bh, ah + bg)$$

(a,b)*(g,h) = (ag - bh, ah + bg)

If I have 2 angles, can I build their combined properties from the single angles?



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$$sin(\alpha+\beta) = a*h + b*g$$

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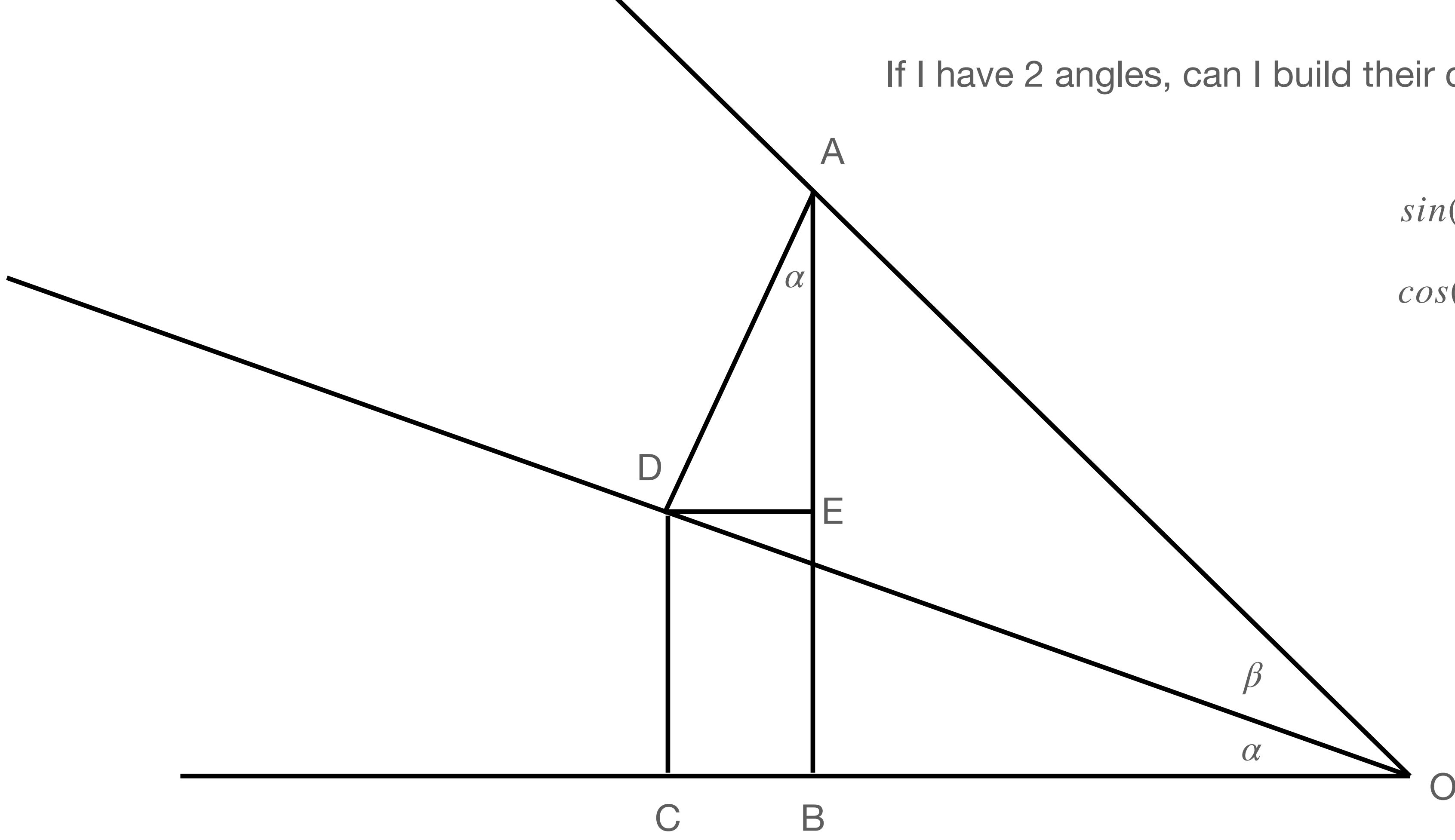
Multiplying complex numbers is somehow like adding angles

If I have 2 angles, can I build their combined properties from the single angles?



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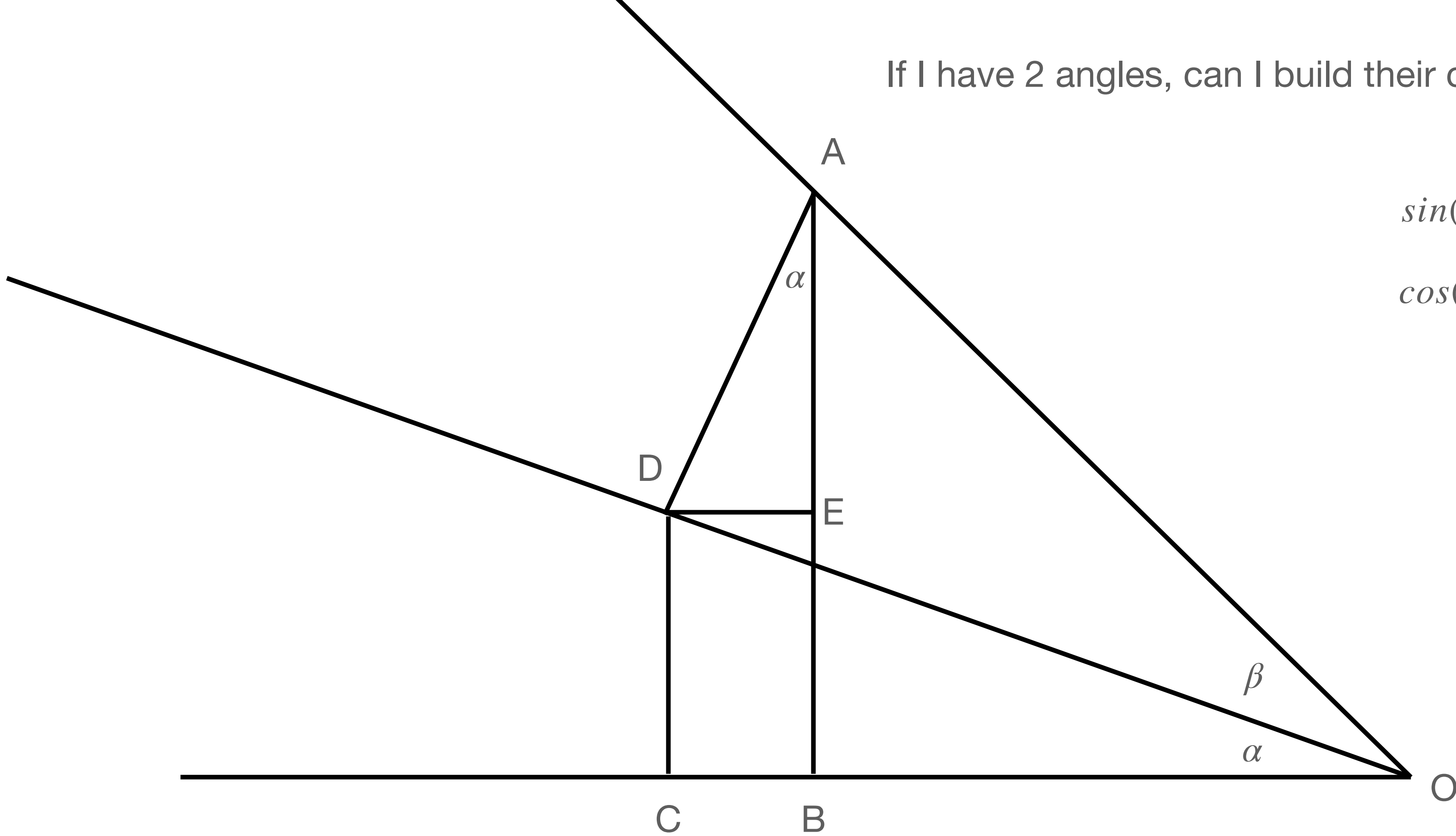
$$(cos(\alpha + \beta), sin(\alpha + \beta)) = (ag - bh, ah + bg)$$

(a,b)*(g,h) = (ag - bh, ah + bg)

Multiplying complex numbers is somehow like adding angles **as if**

If I have 2 angles, can I build their combined properties from the single angles?



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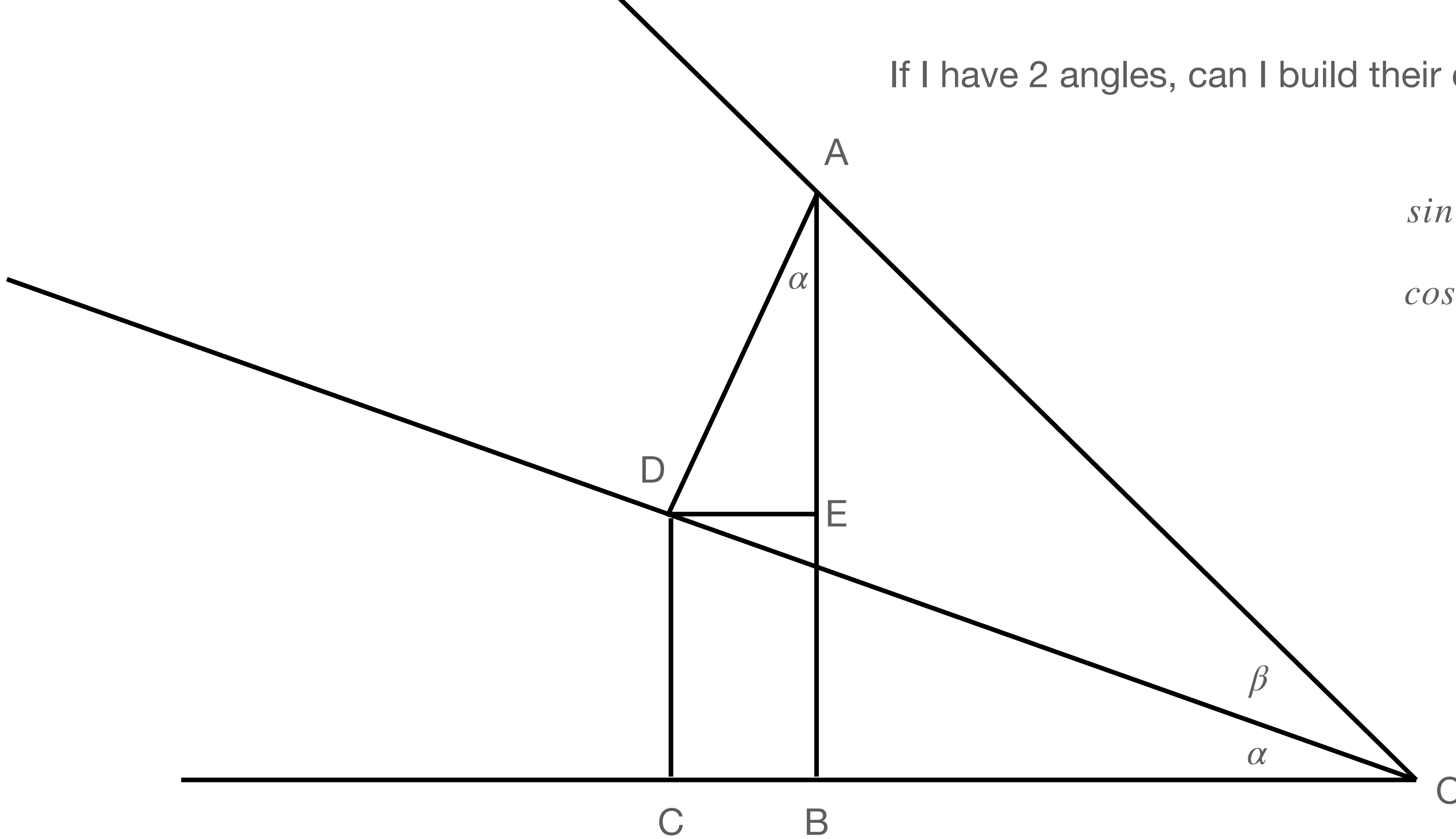
$g = cos(\beta)$ $h = sin(\beta)$ $(g, h) = (cos(\beta), sin(\beta))$

$$(cos(\alpha + \beta), sin(\alpha + \beta)) = (ag - bh, ah + bg)$$

(a,b)*(g,h) = (ag - bh, ah + bg)

Multiplying complex numbers is somehow like adding angles **as if** **(a,b) were (cos(x),sin(x))**

If I have 2 angles, can I build their combined properties from the single angles?



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$a = cos(\alpha)$ $b = sin(\alpha)$ $(a, b) = (cos(\alpha), sin(\alpha))$

$g = cos(\beta)$ $h = sin(\beta)$ $(g, h) = (cos(\beta), sin(\beta))$

$$(cos(\alpha+\beta), sin(\alpha+\beta)) = (ag - bh, ah + bg)$$

(a,b)*(g,h) = (ag - bh, ah + bg)

Multiplying complex numbers is somehow like adding angles **as if** **(a,b) were (cos(x),sin(x))**
(g,h) were (cos(y),sin(y))

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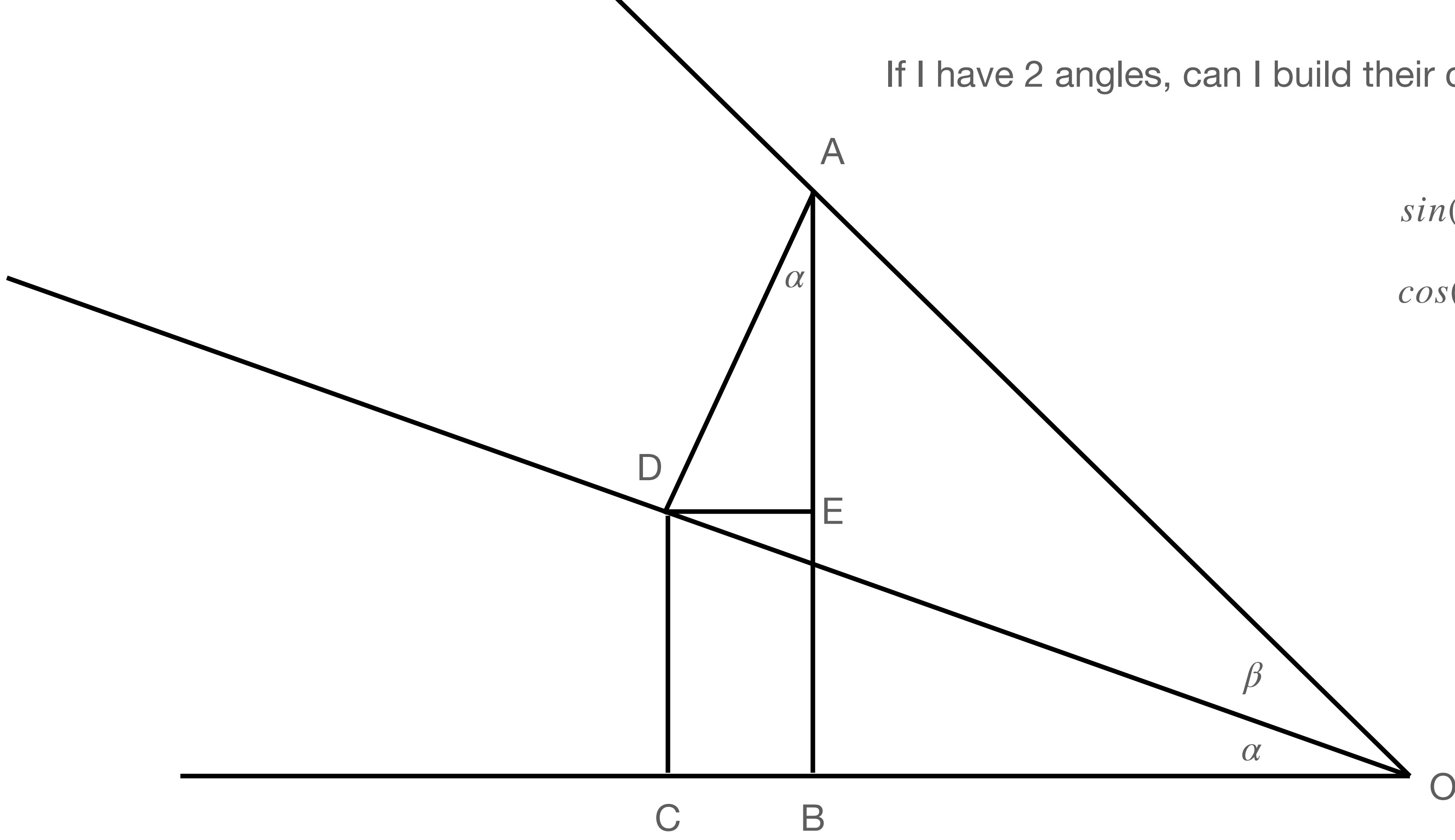

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Multiplying complex numbers is somehow like adding angles **as if** **(a,b) were (cos(x),sin(x))** **and** **(g,h) were (cos(y),sin(y))**

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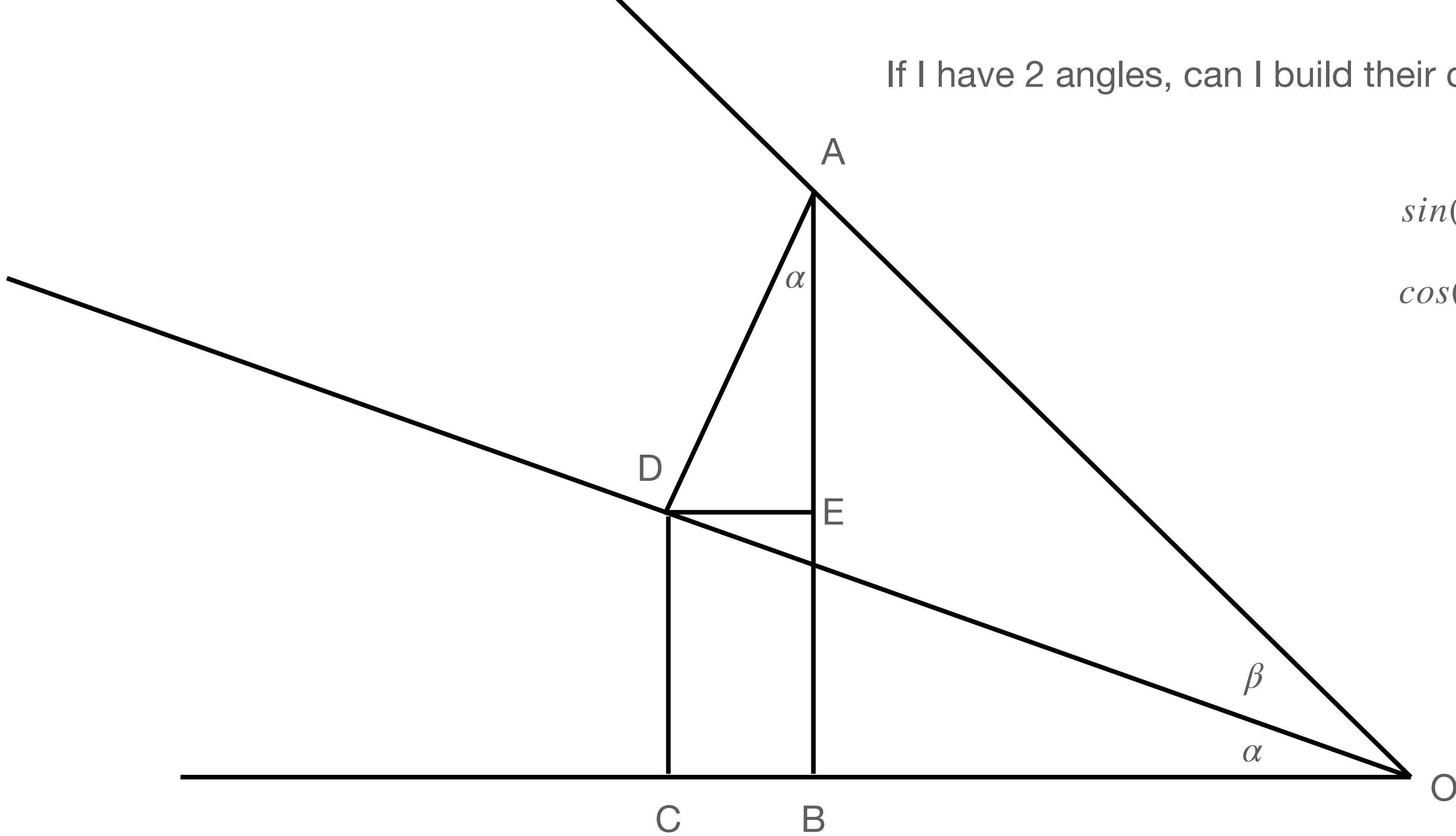

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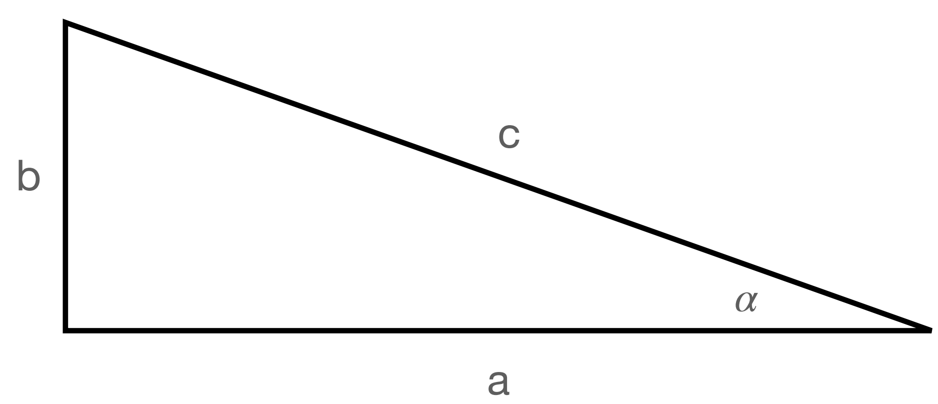
Multiplying complex numbers is somehow like adding angles **as if** **(a,b) were (cos(x),sin(x))** **(g,h) were (cos(y),sin(y))** **and** **(a,b)*(x,y)=(cos(x+y),sin(x+y)**

In a right triangle, by definition



$$sin(\alpha) = \frac{opp}{hyp} = \frac{b}{c}$$

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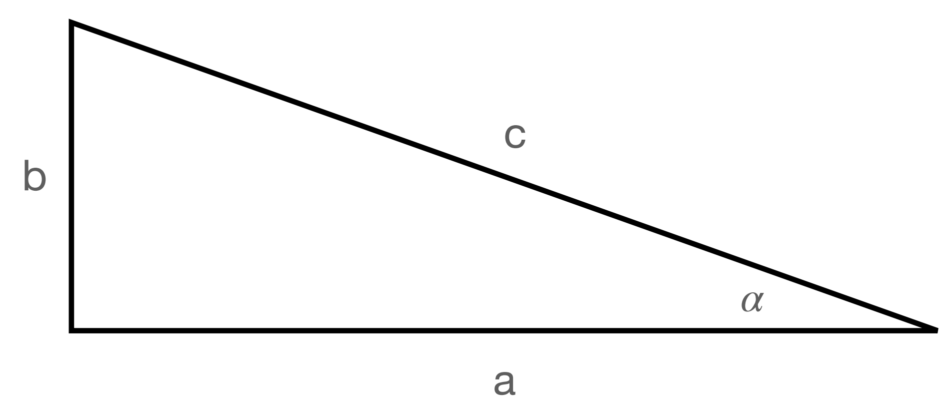
In a right triangle, by definition



$$sin(\alpha) = \frac{opp}{hyp} = \frac{b}{c}$$

$$b = c * sin(\alpha)$$

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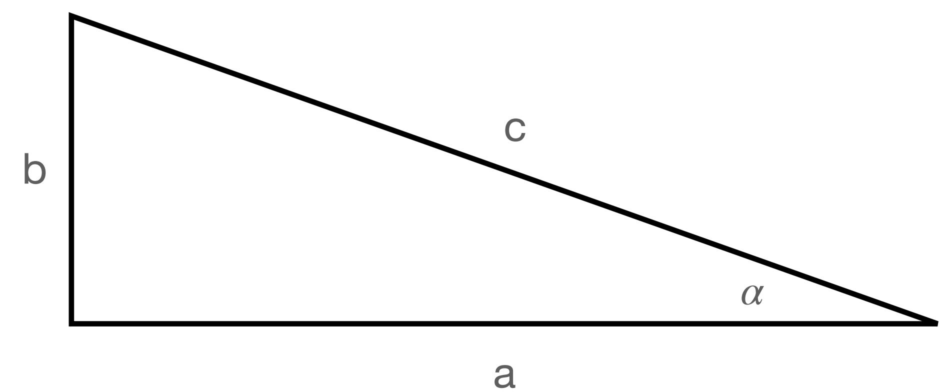

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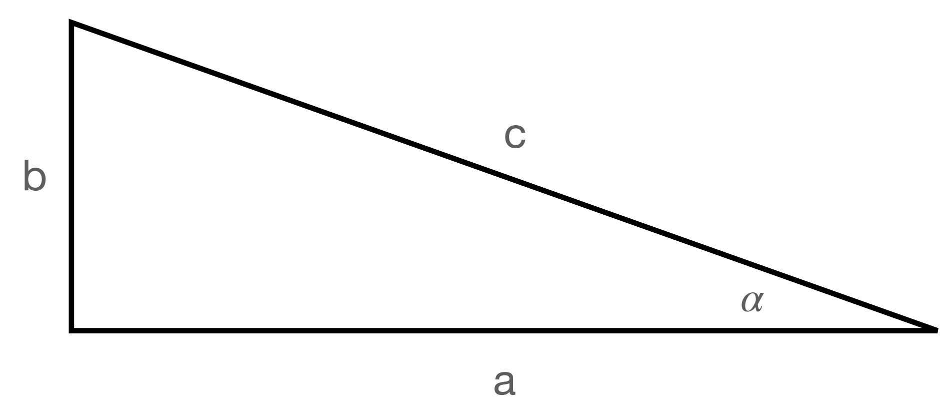

c is a scaling factor here

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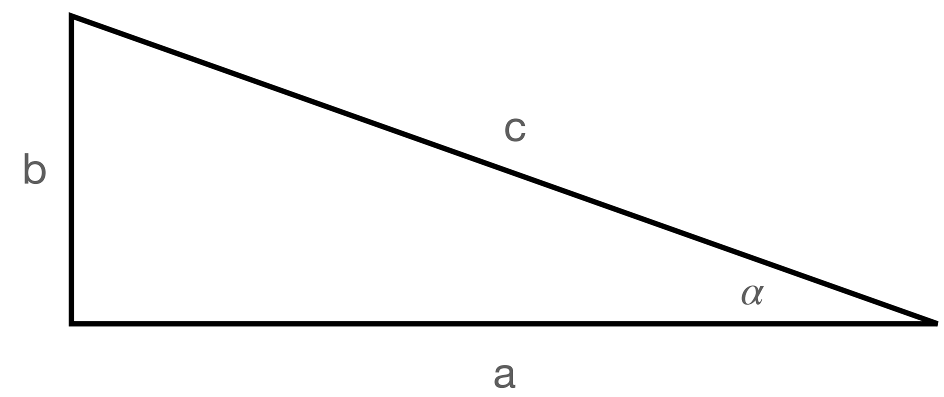

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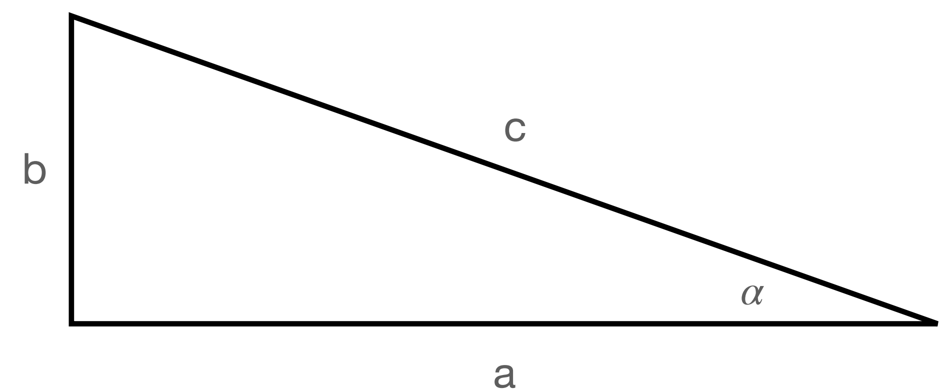

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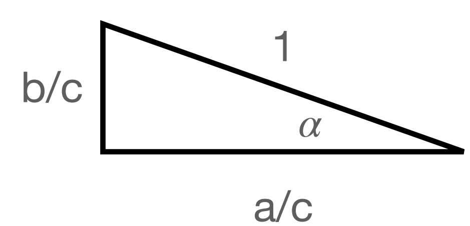

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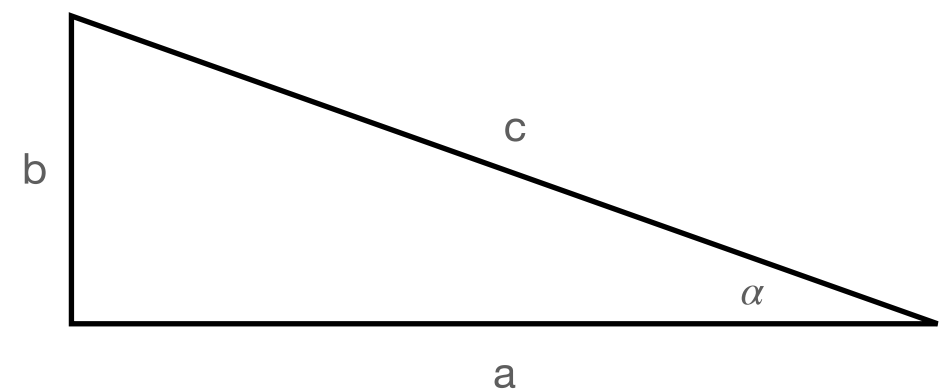

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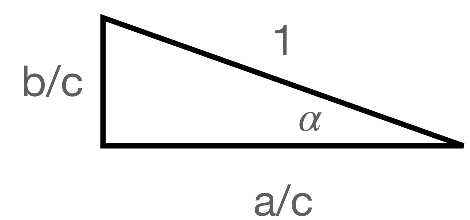

$$sin(\alpha) = \frac{b}{c} = \frac{b/c}{c/c} = \frac{b/c}{1} = \frac{opp}{hyp}$$

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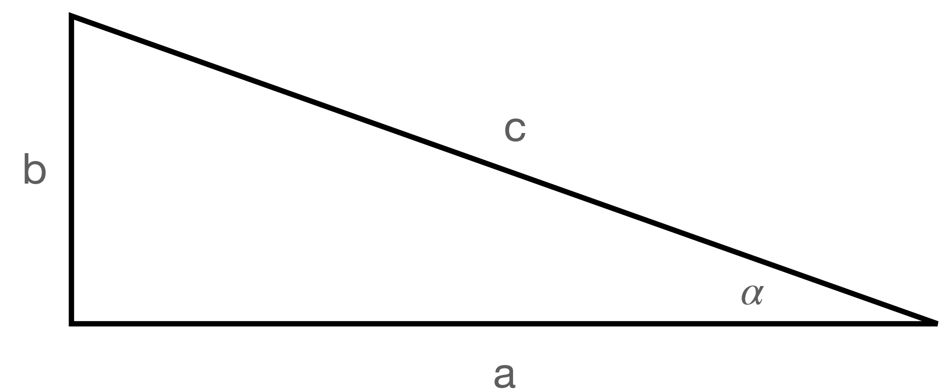

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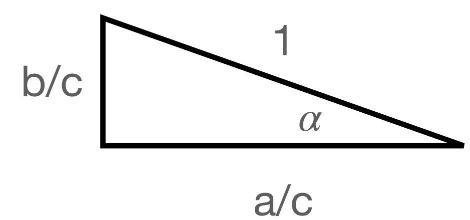

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From here on, we use unit triangles and the unit circle, to reduce nomenclature. Because we are concerned with angles, not size.

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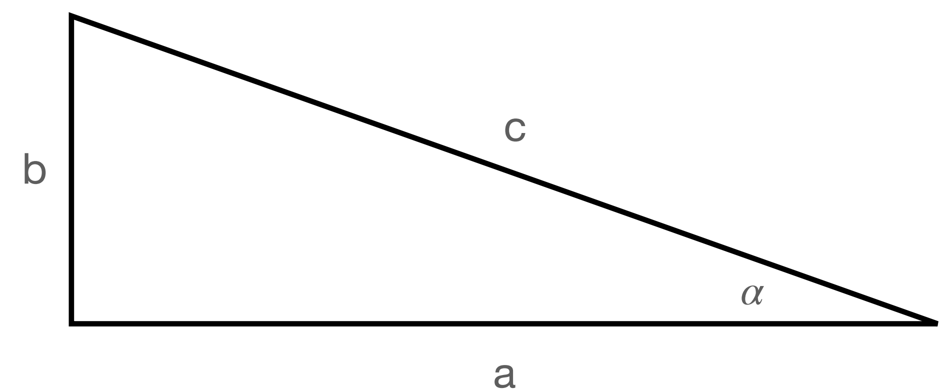

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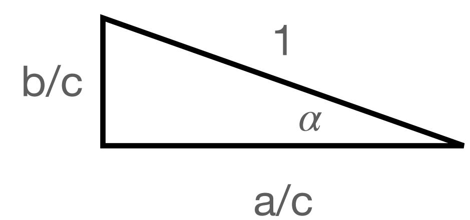

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From here on, we use unit triangles and the unit circle, to reduce nomenclature. Because we are concerned with angles, not size.

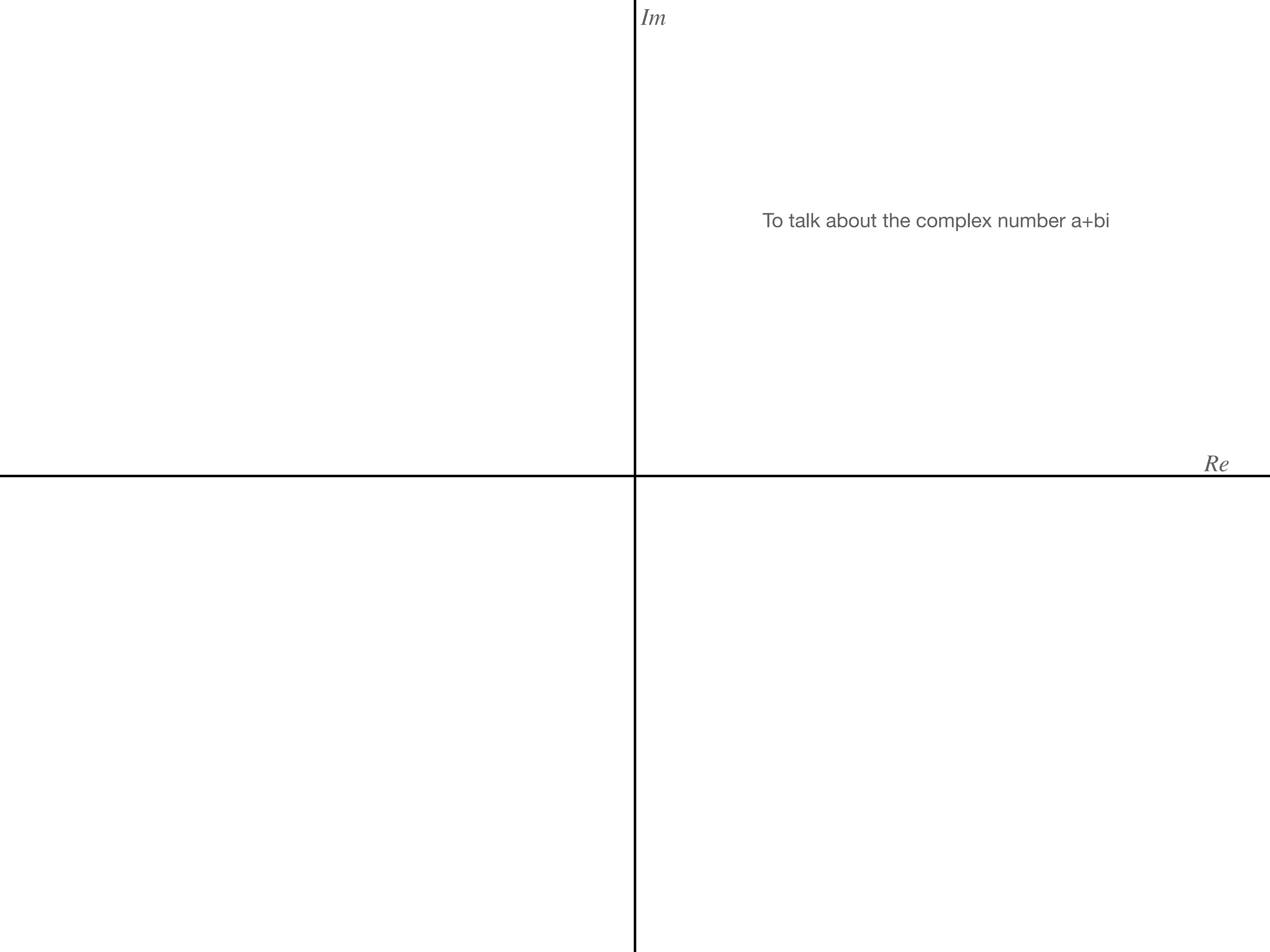
But, to do work with arbitrary complex (a,b) viewed as angles, first we show:

1. That (a,b) can be part of a right triangle
2. That we can get c and alpha from (a,b) right triangles
3. That we can normalize triangles to unit length and correct sizes later

To talk about the complex number a+bi

To talk about the complex number a+bi

Re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To talk about the complex number a+bi
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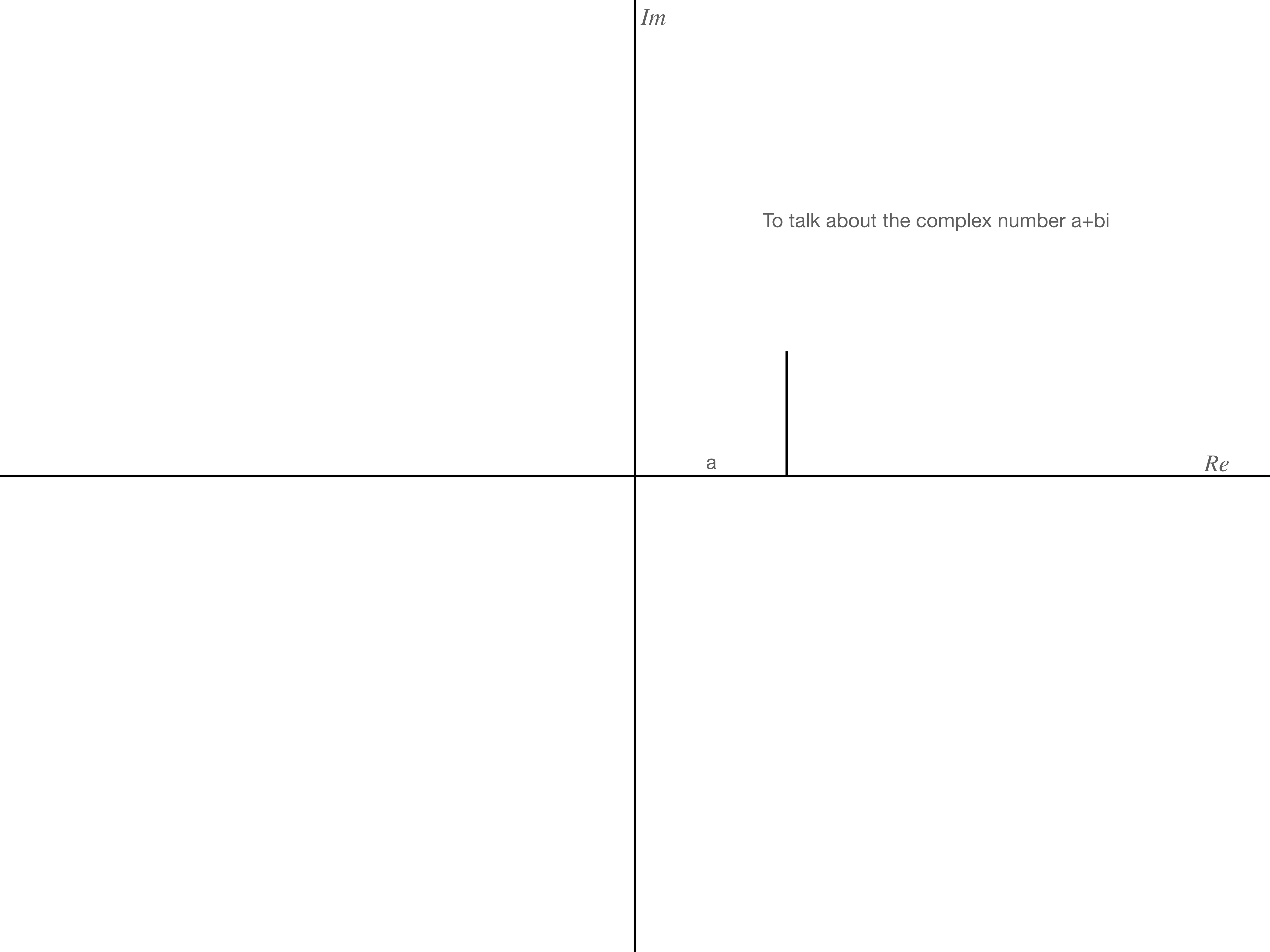
Im

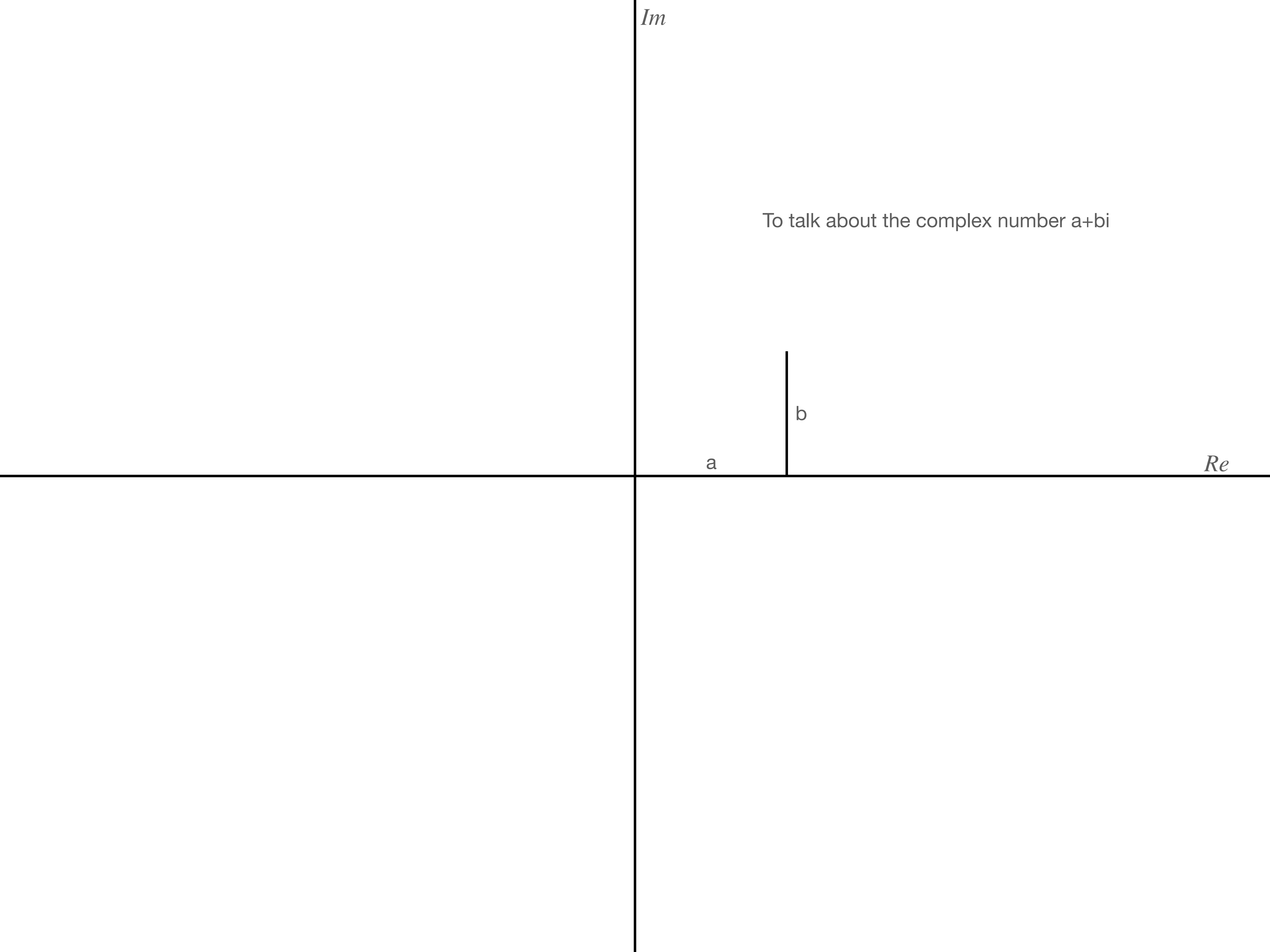
To talk about the complex number a+bi

a

Re

To talk about the complex number a+bi

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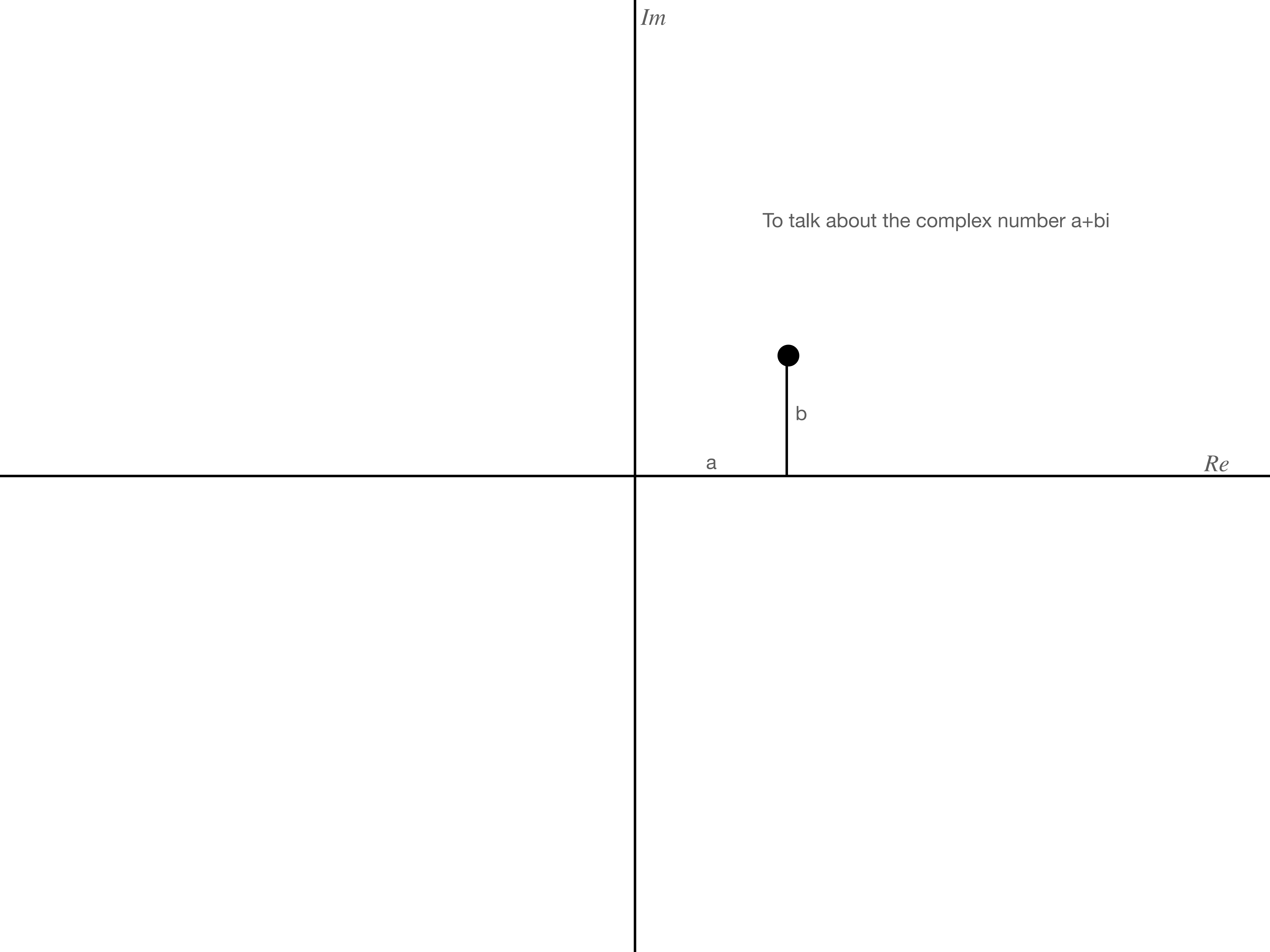
a
Re


Im
To talk about the complex number a+bi
b
a
Re

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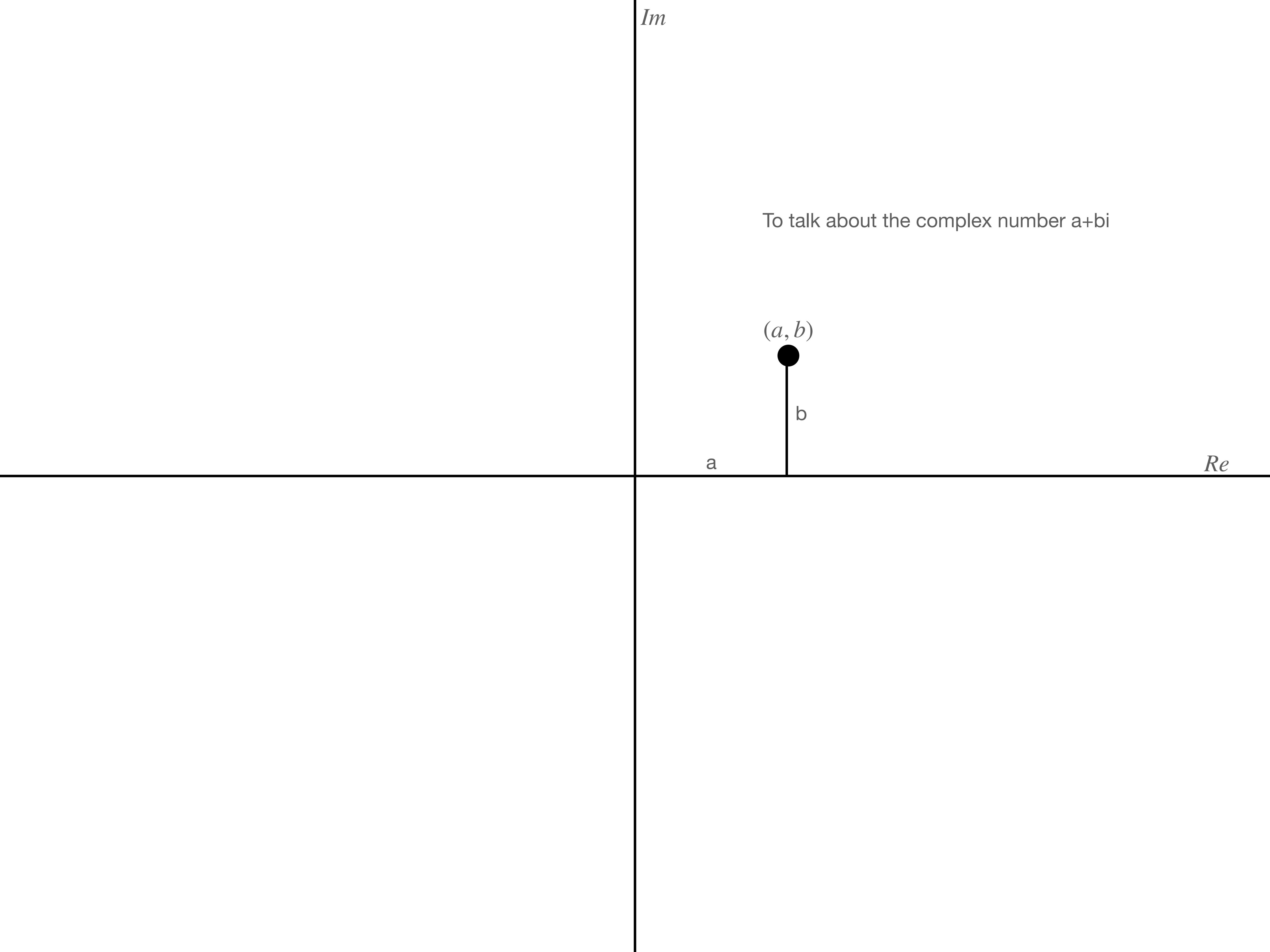
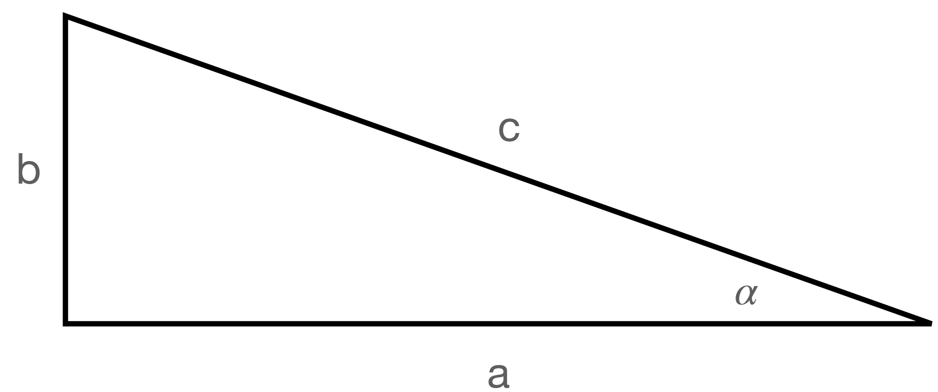


In a right triangle, by definition



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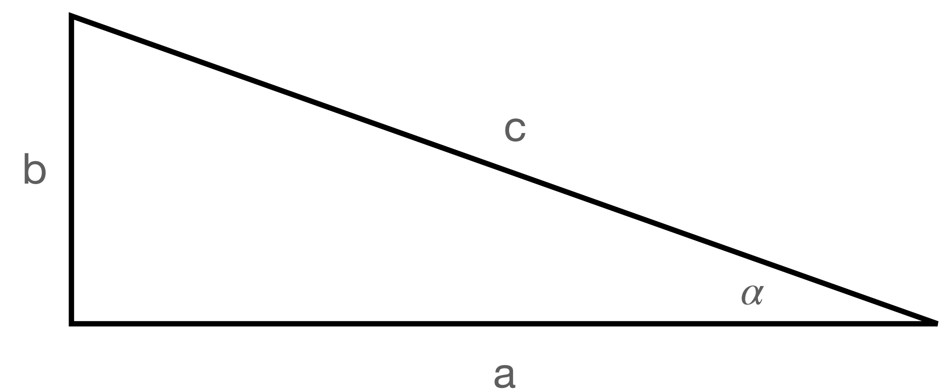
2: If you have imaginary number (a,b)

In a right triangle, by definition



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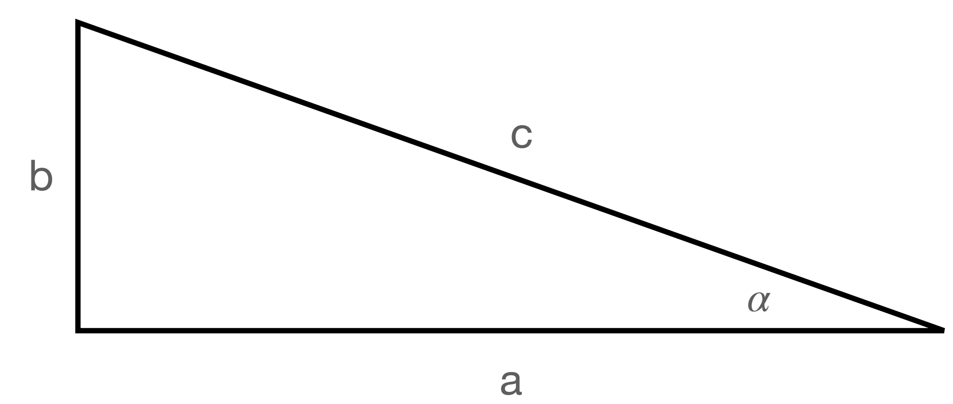

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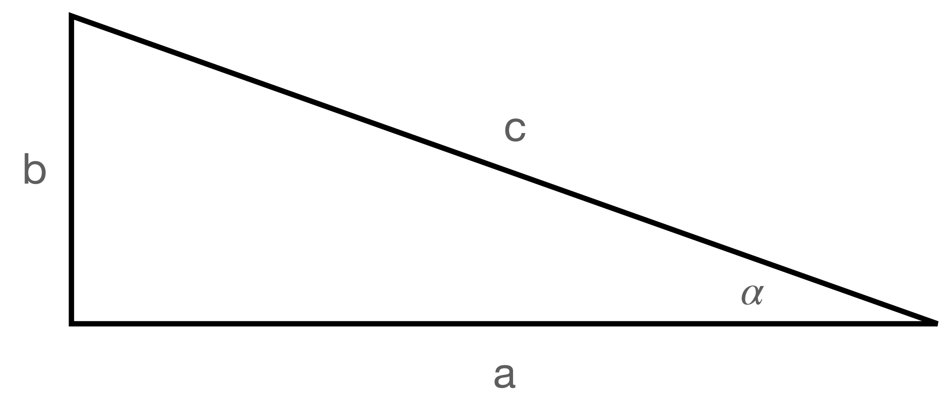

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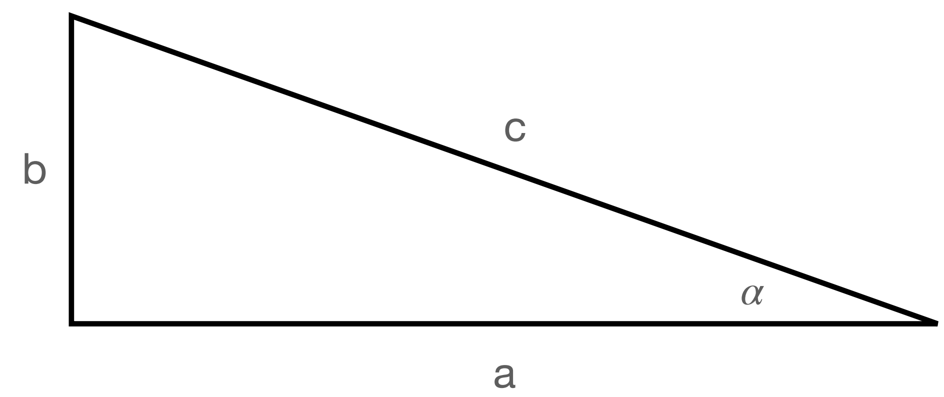

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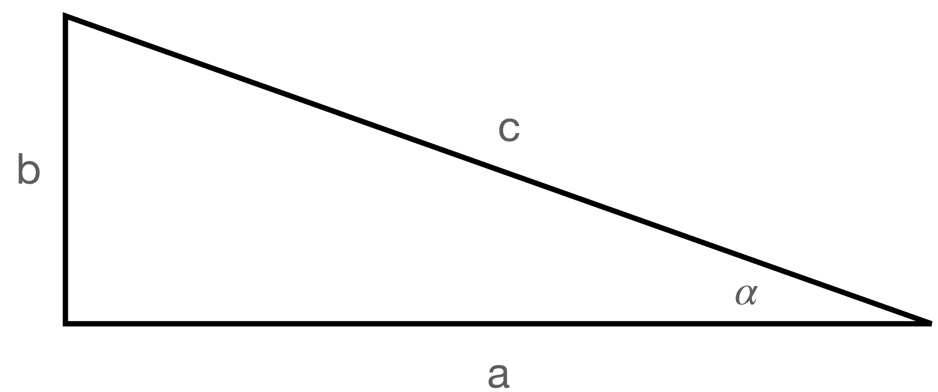

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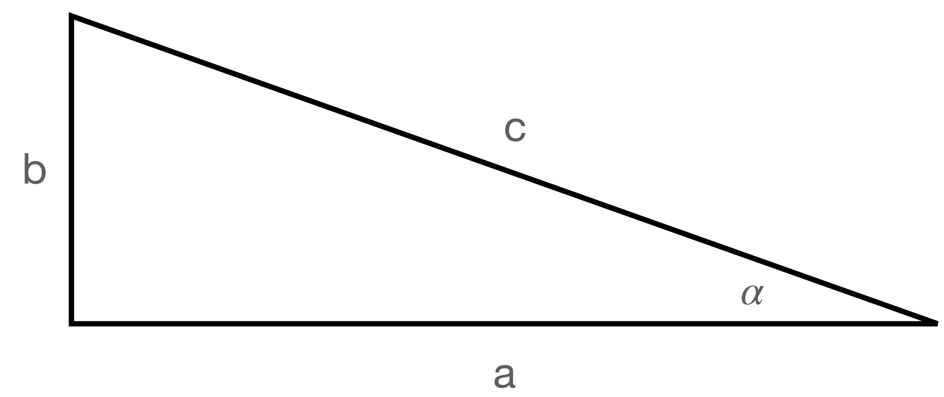

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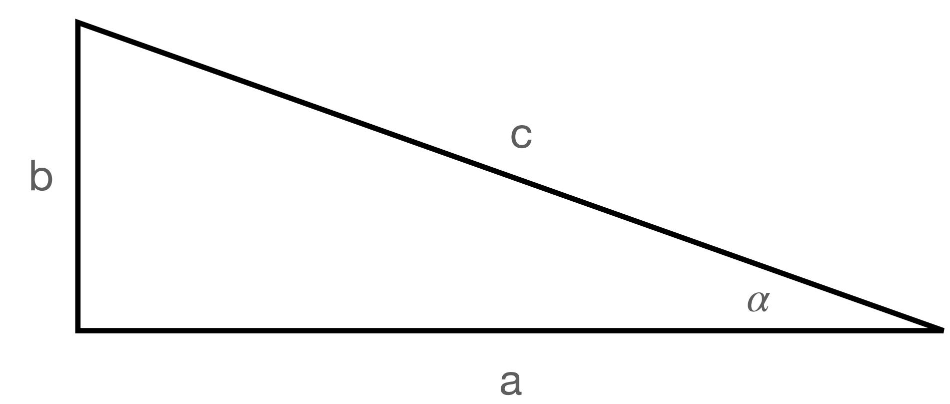

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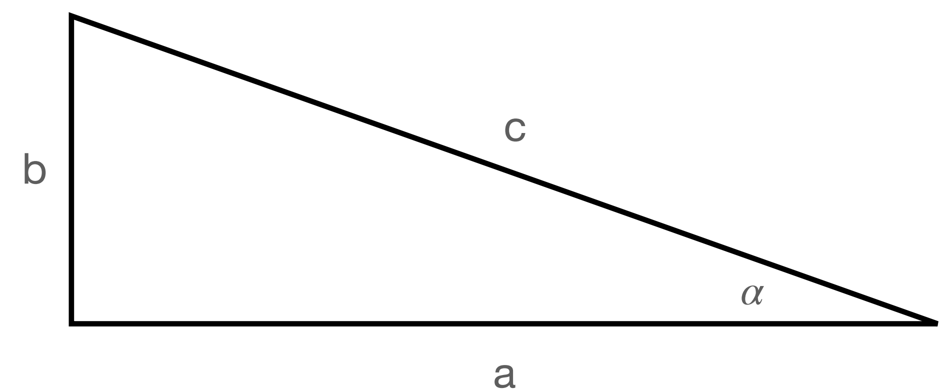

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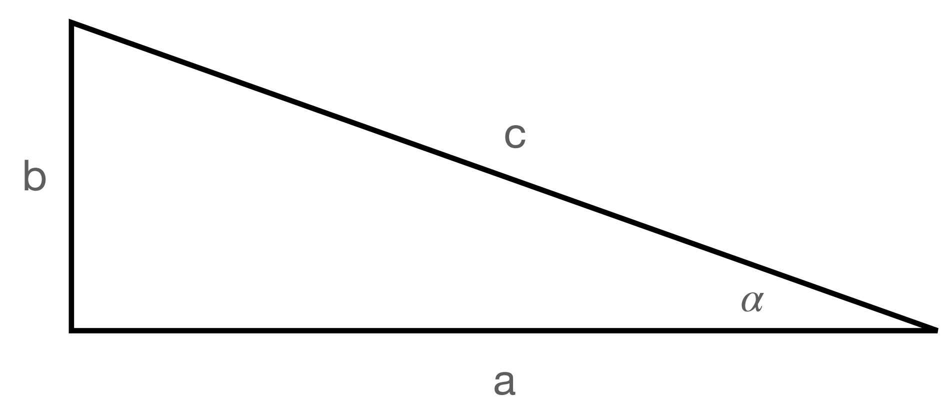

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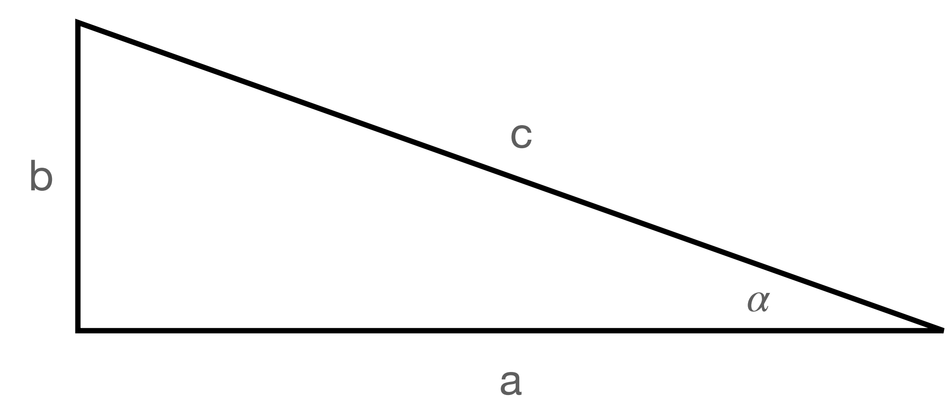

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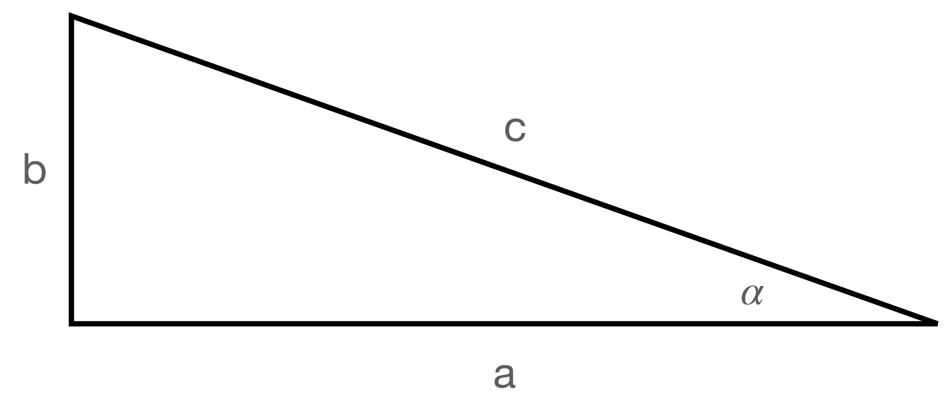

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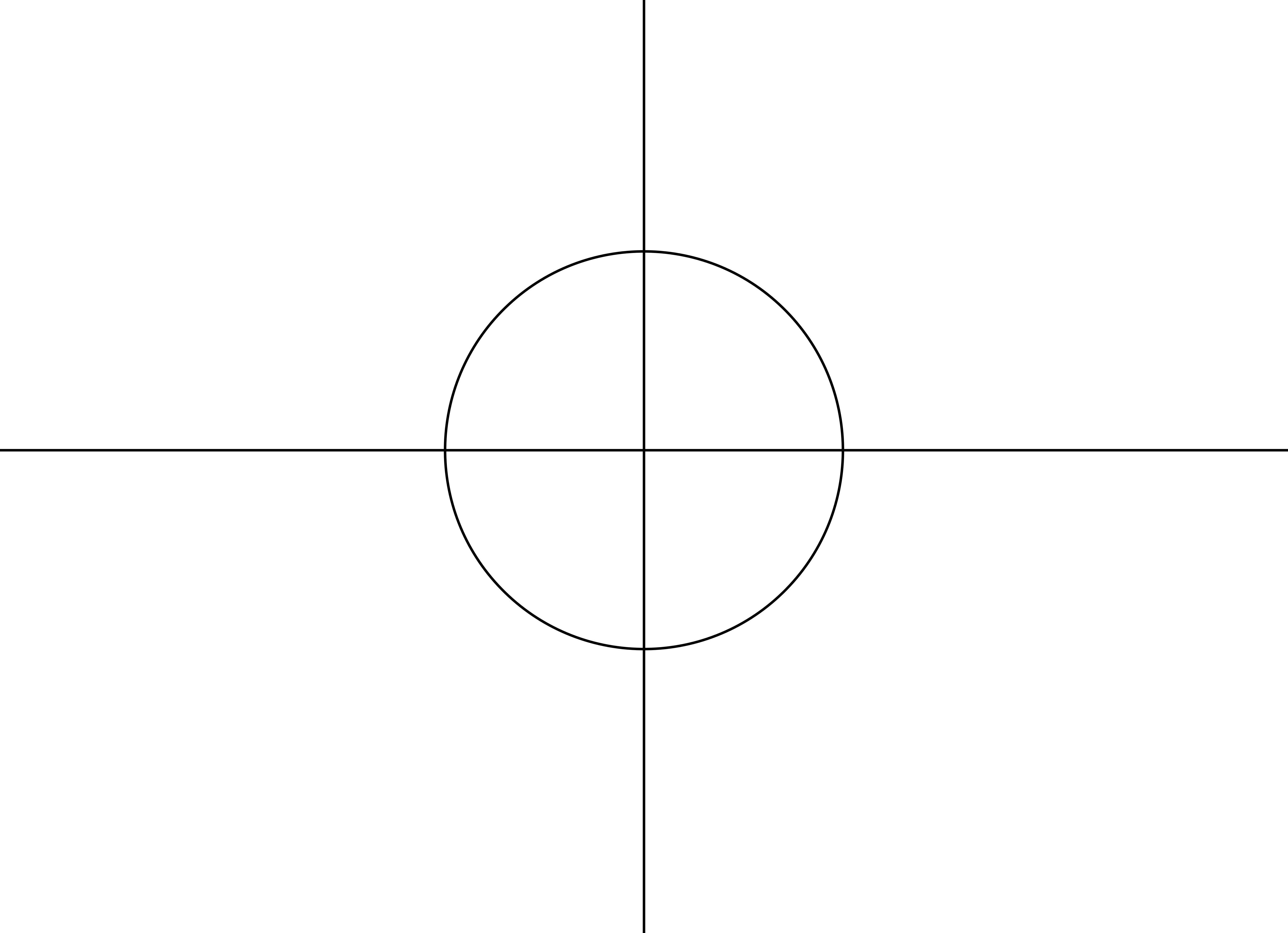
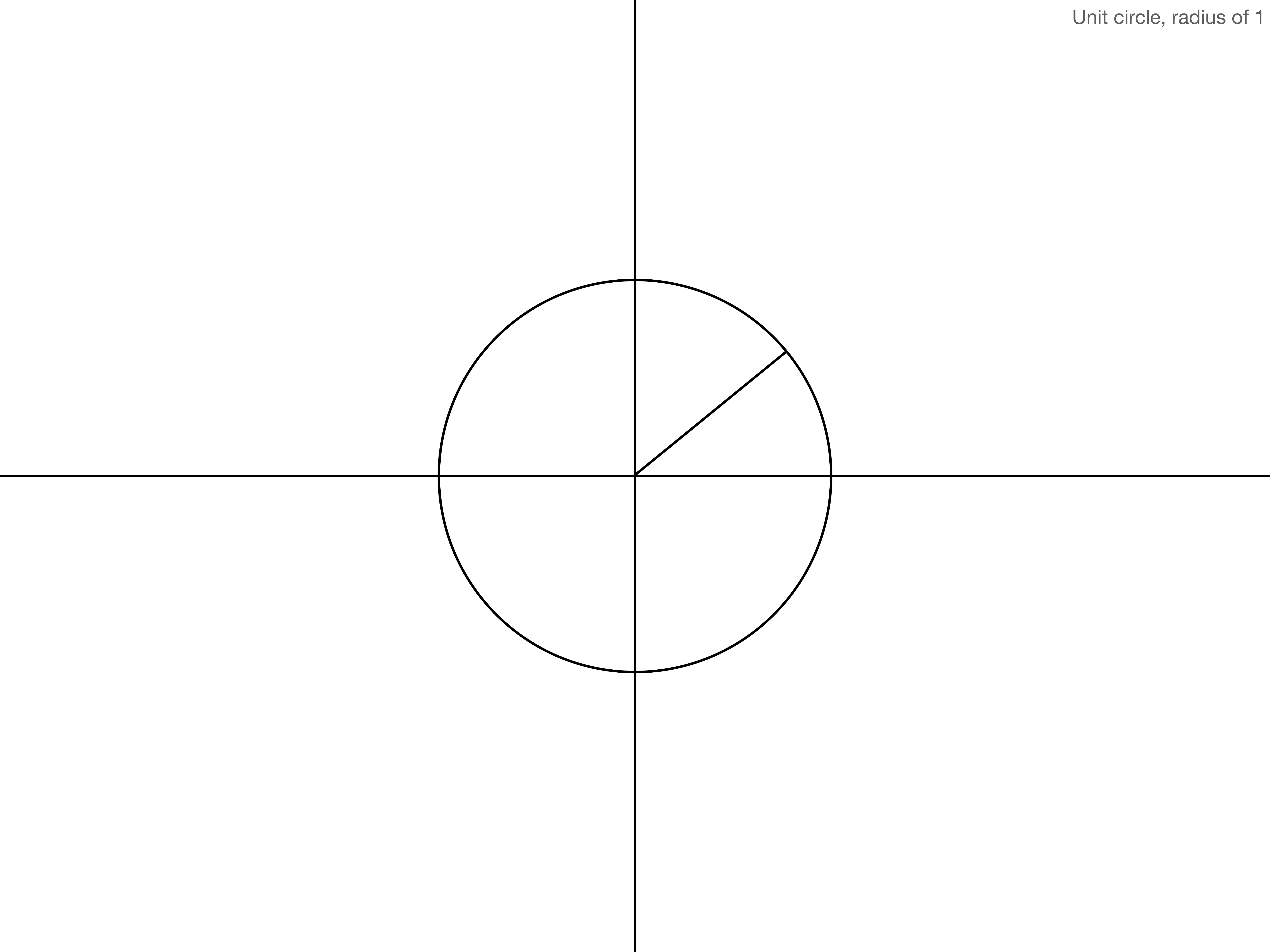
Yup, initial sizes determine final sizes

Unit circle, radius of 1

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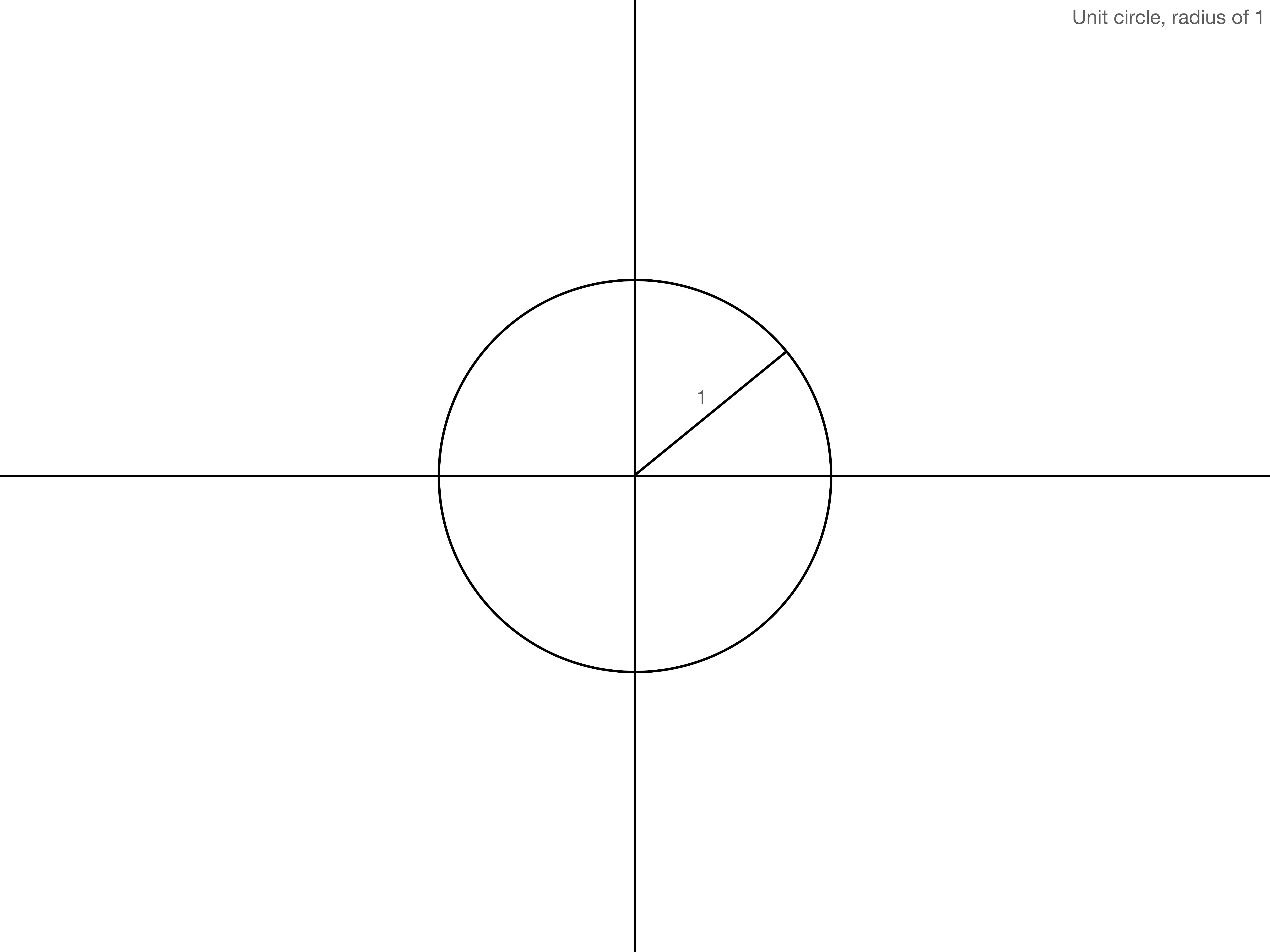

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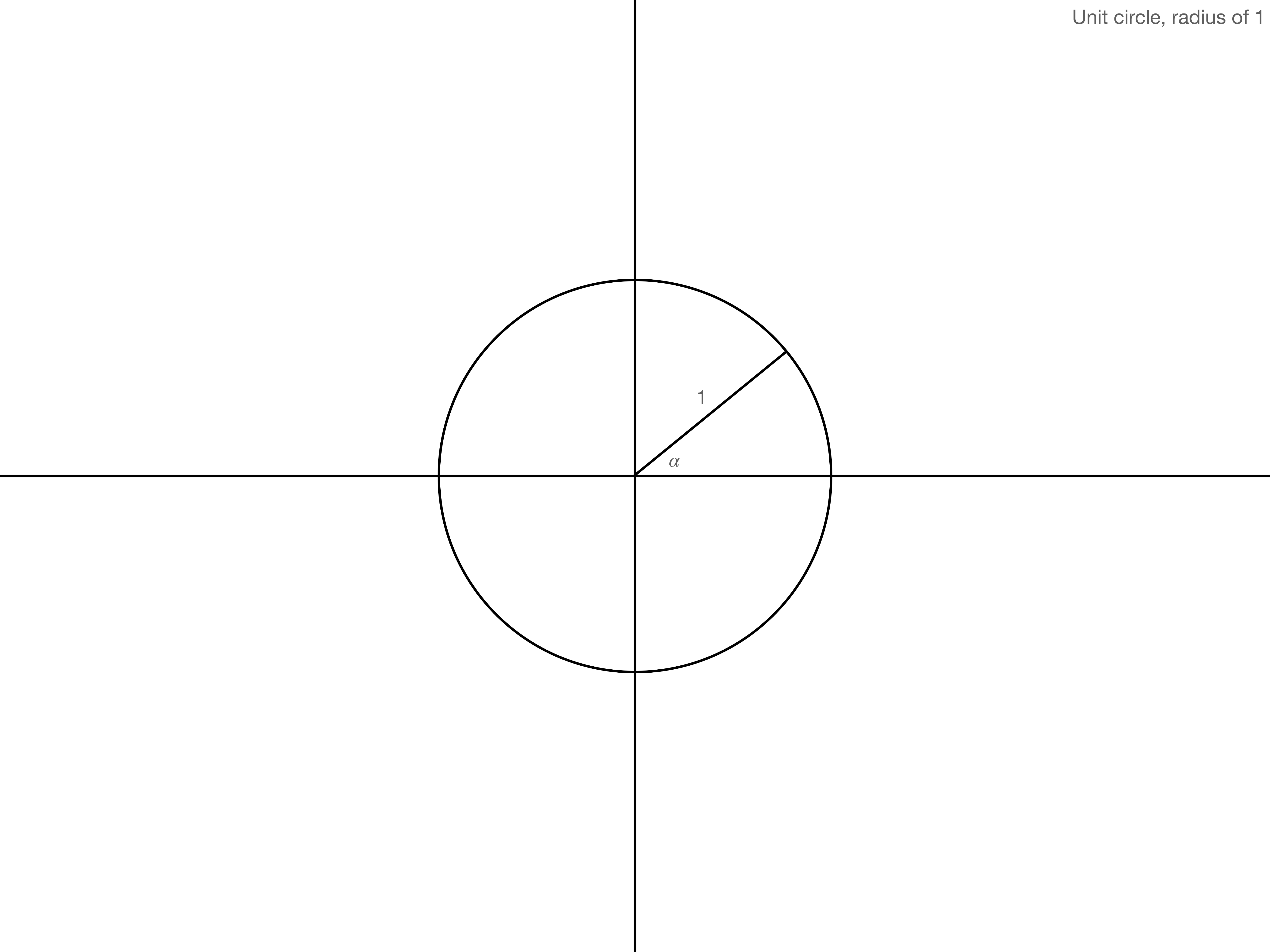

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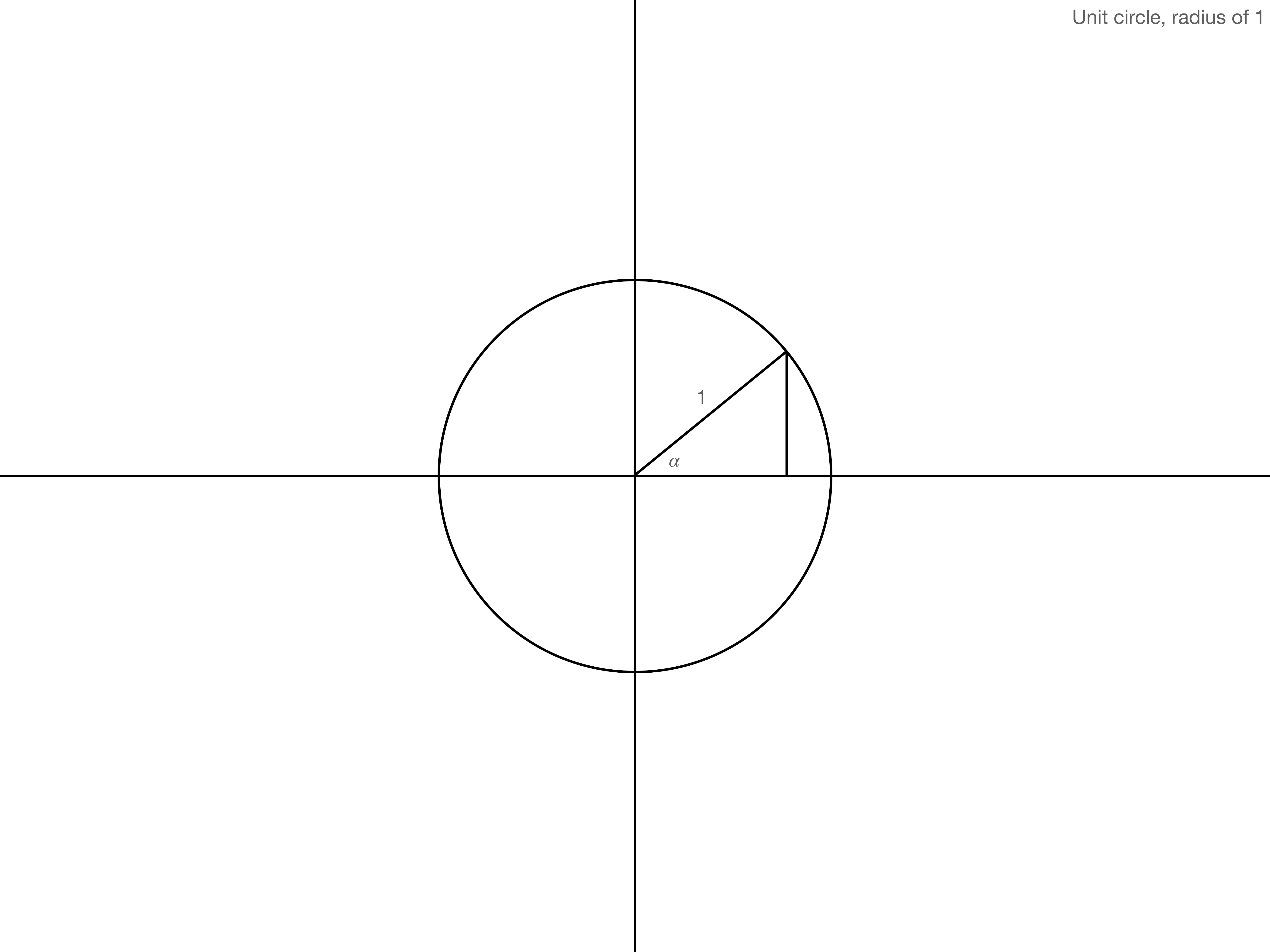

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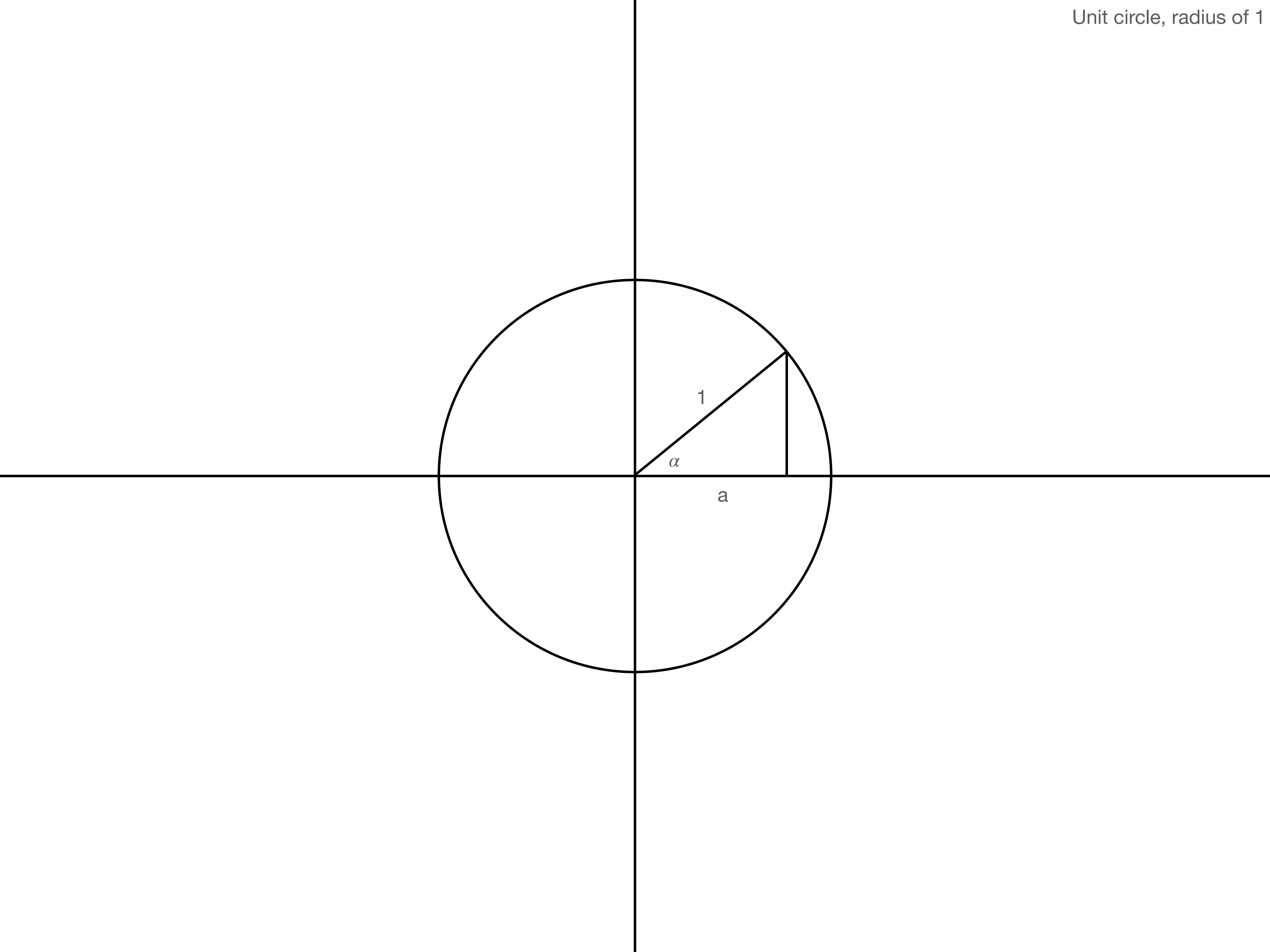

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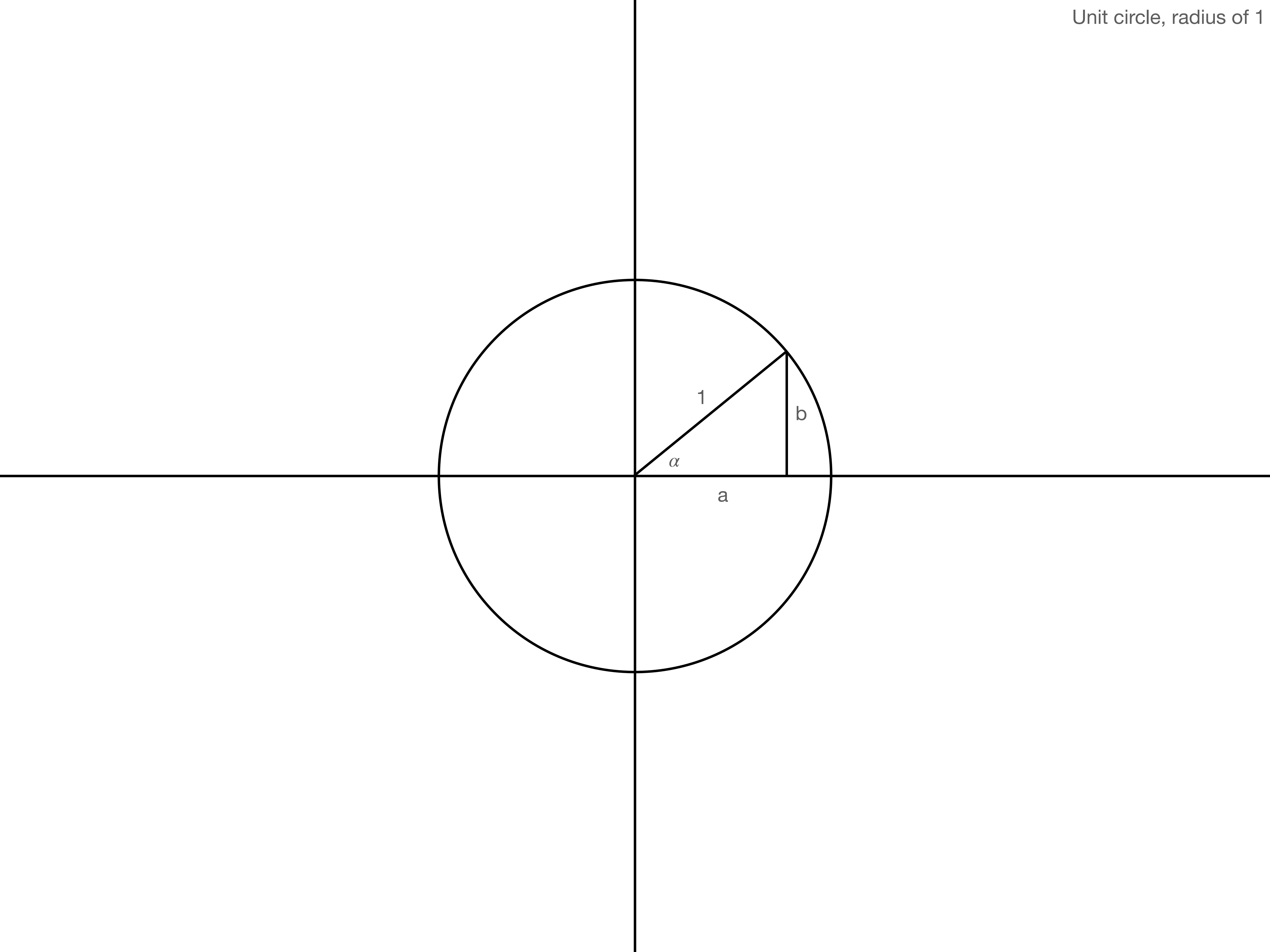

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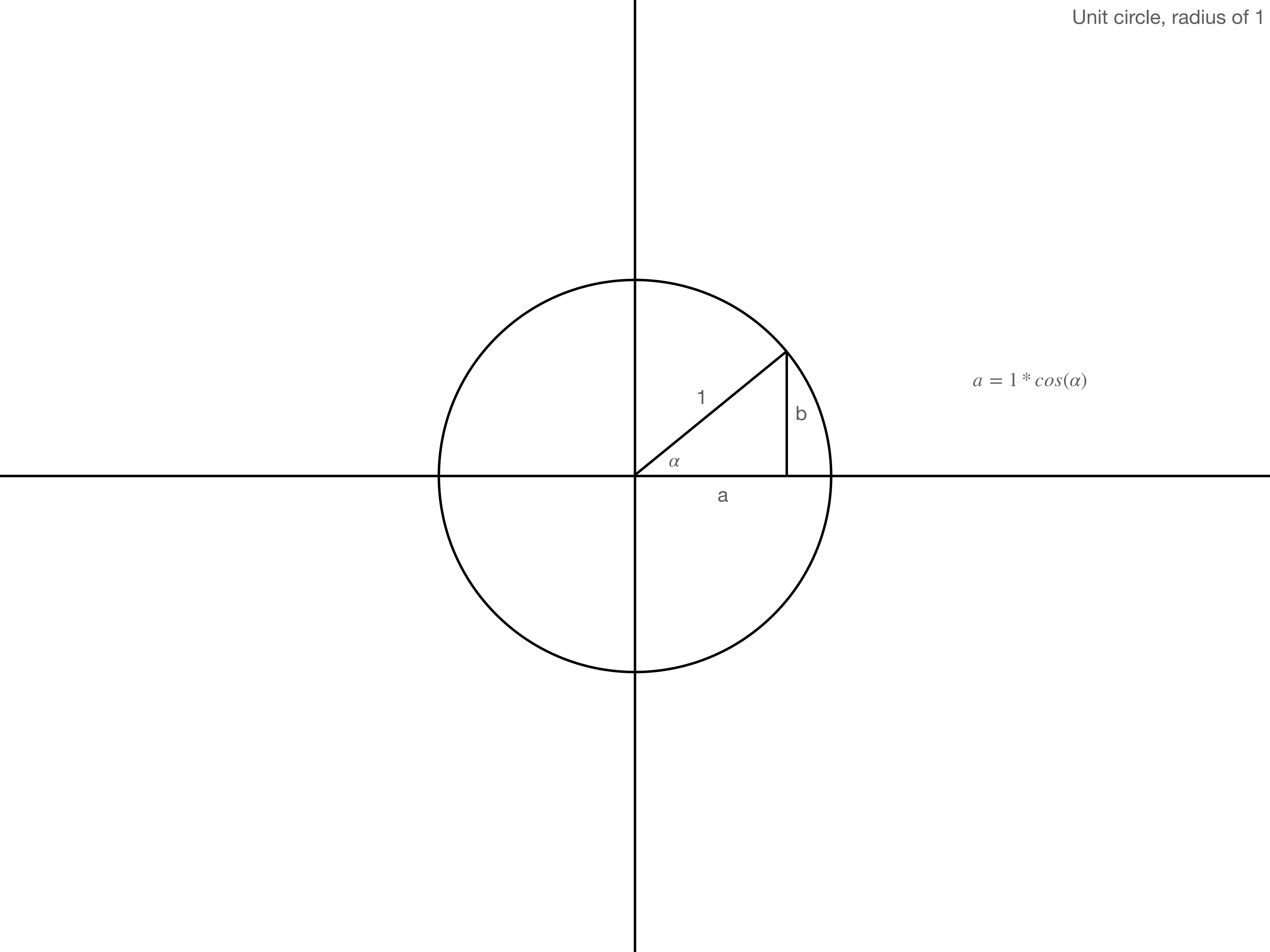
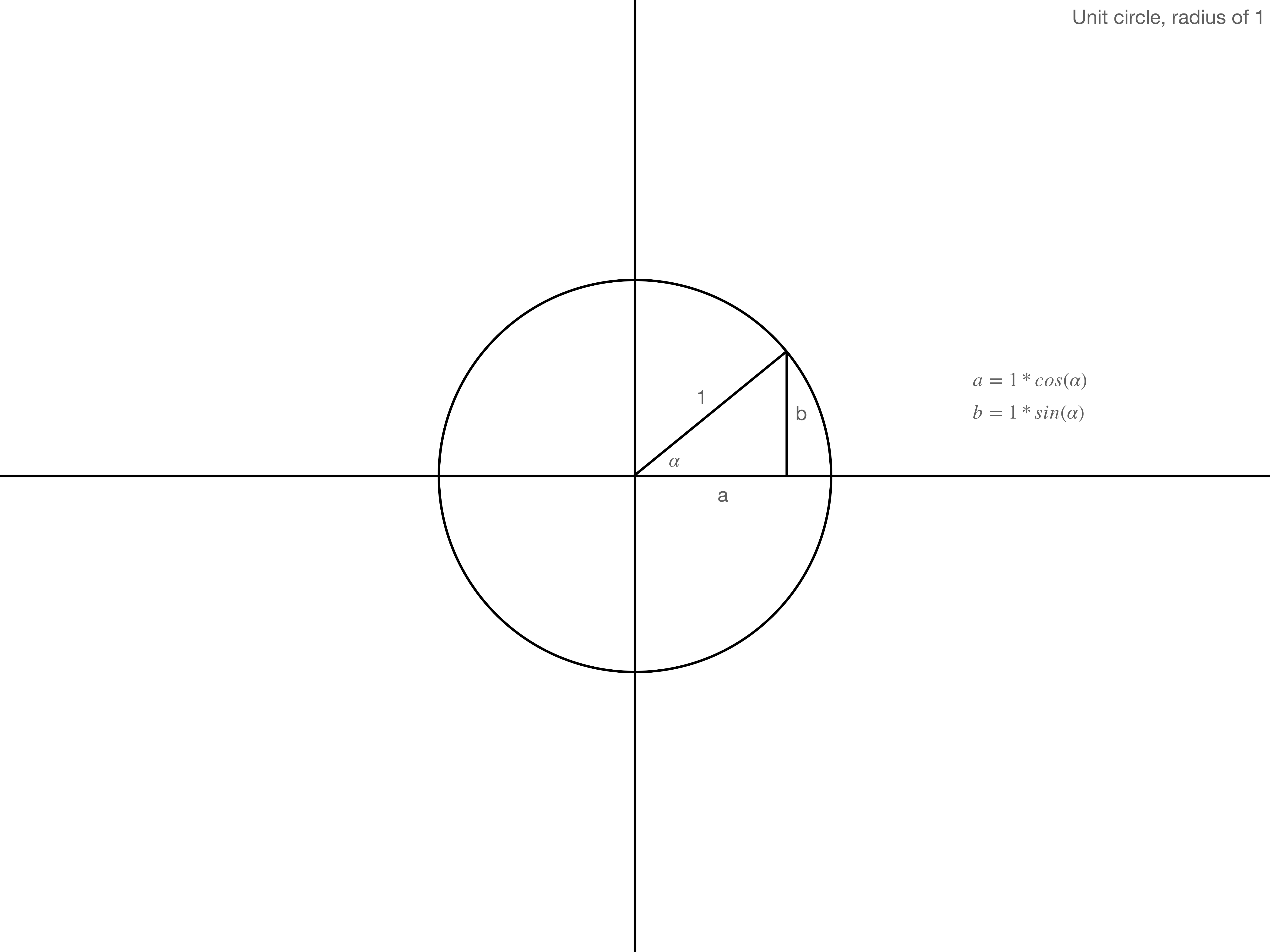


$a = 1 * cos(\alpha)$

Unit circle, radius of 1



$$a = 1 * cos(\alpha)$$

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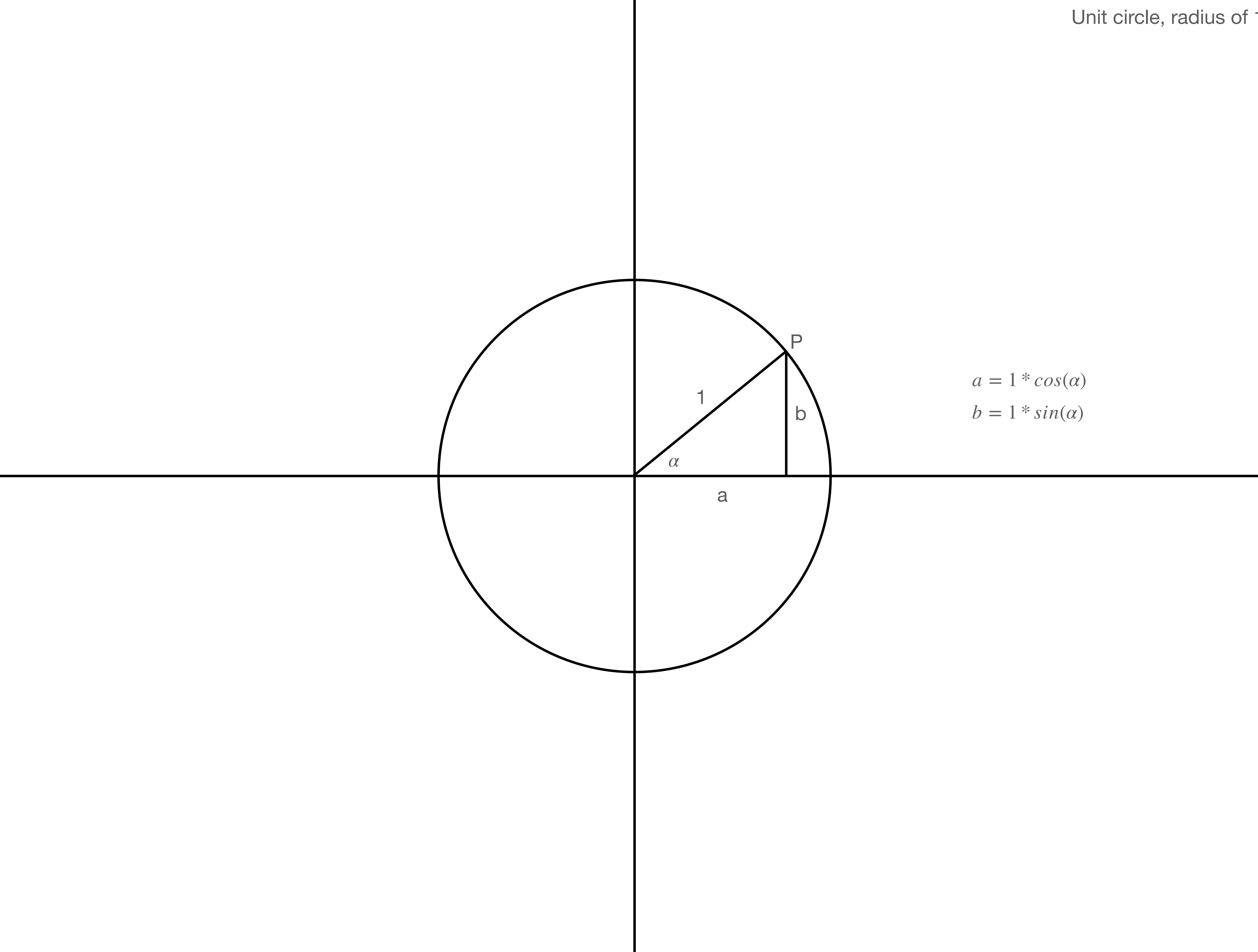
Unit circle, radius of 1



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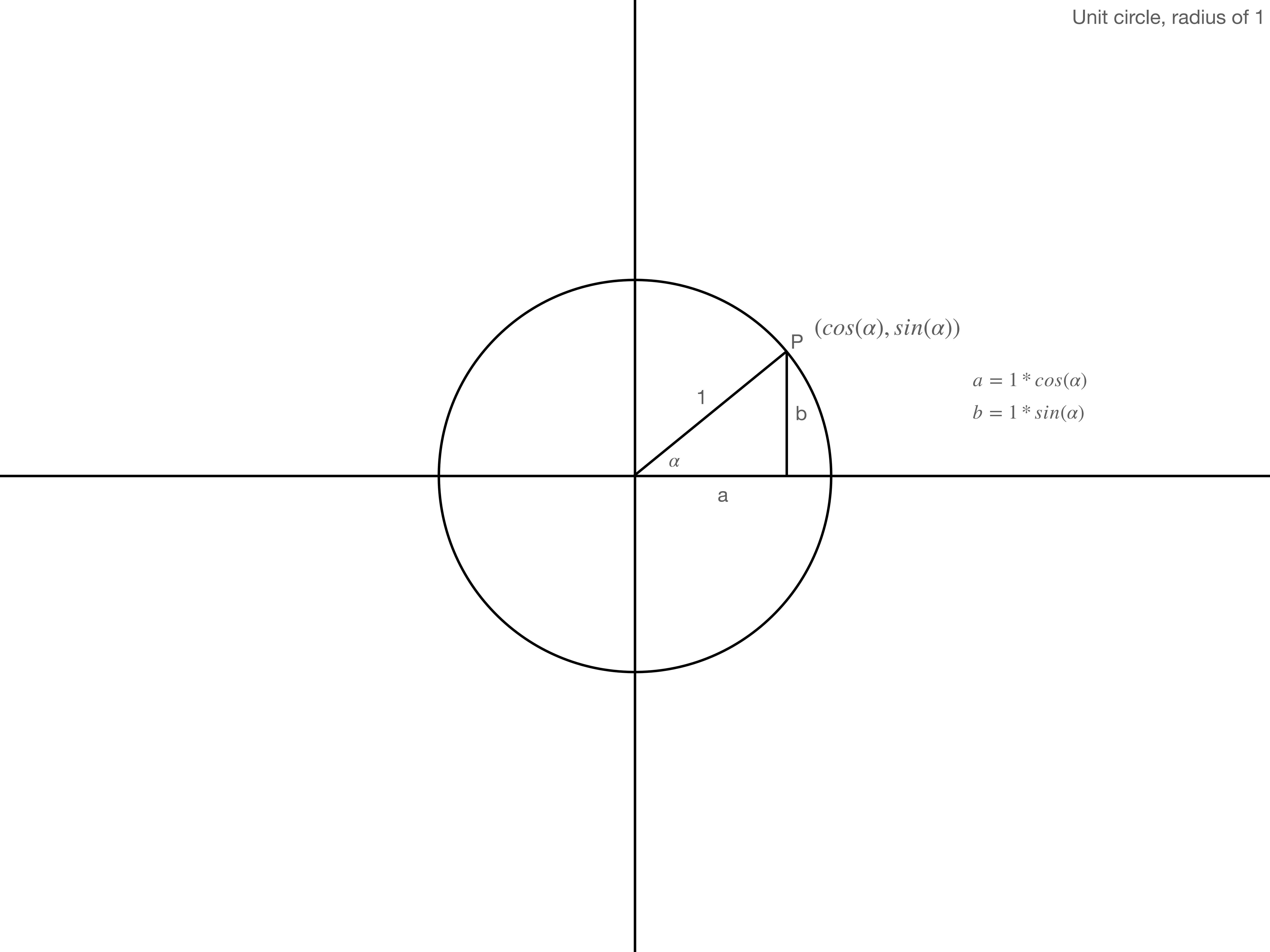

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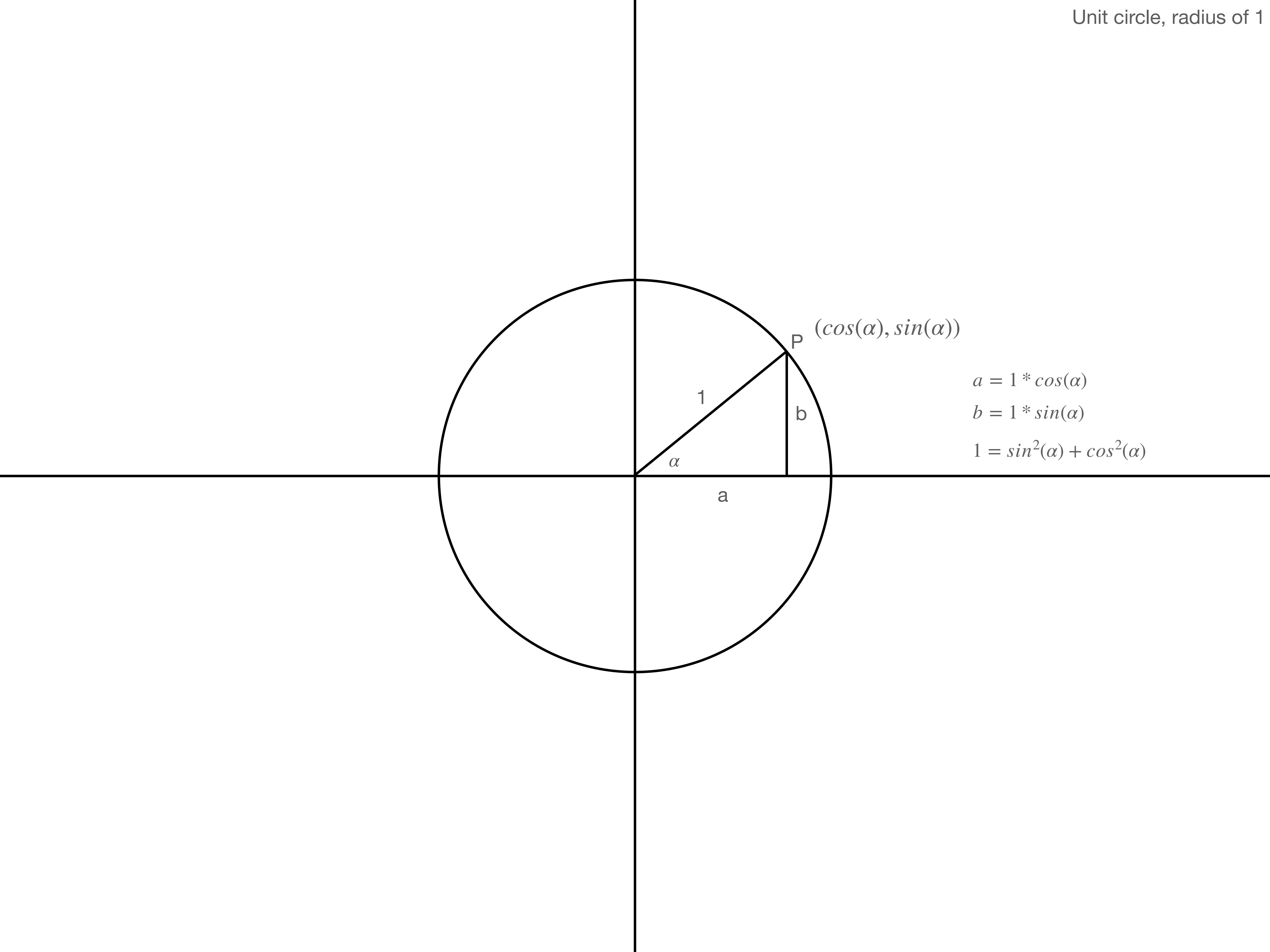

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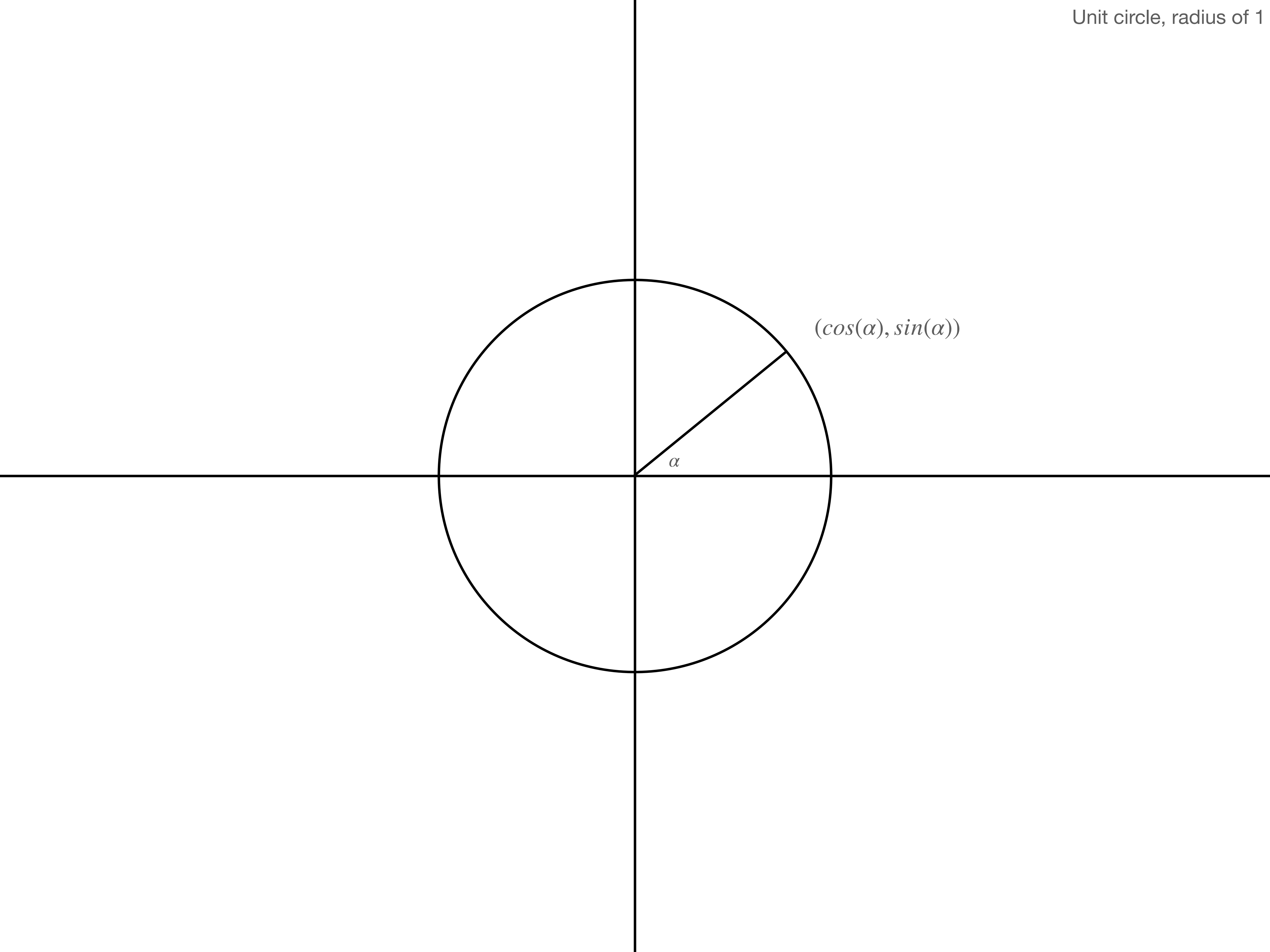
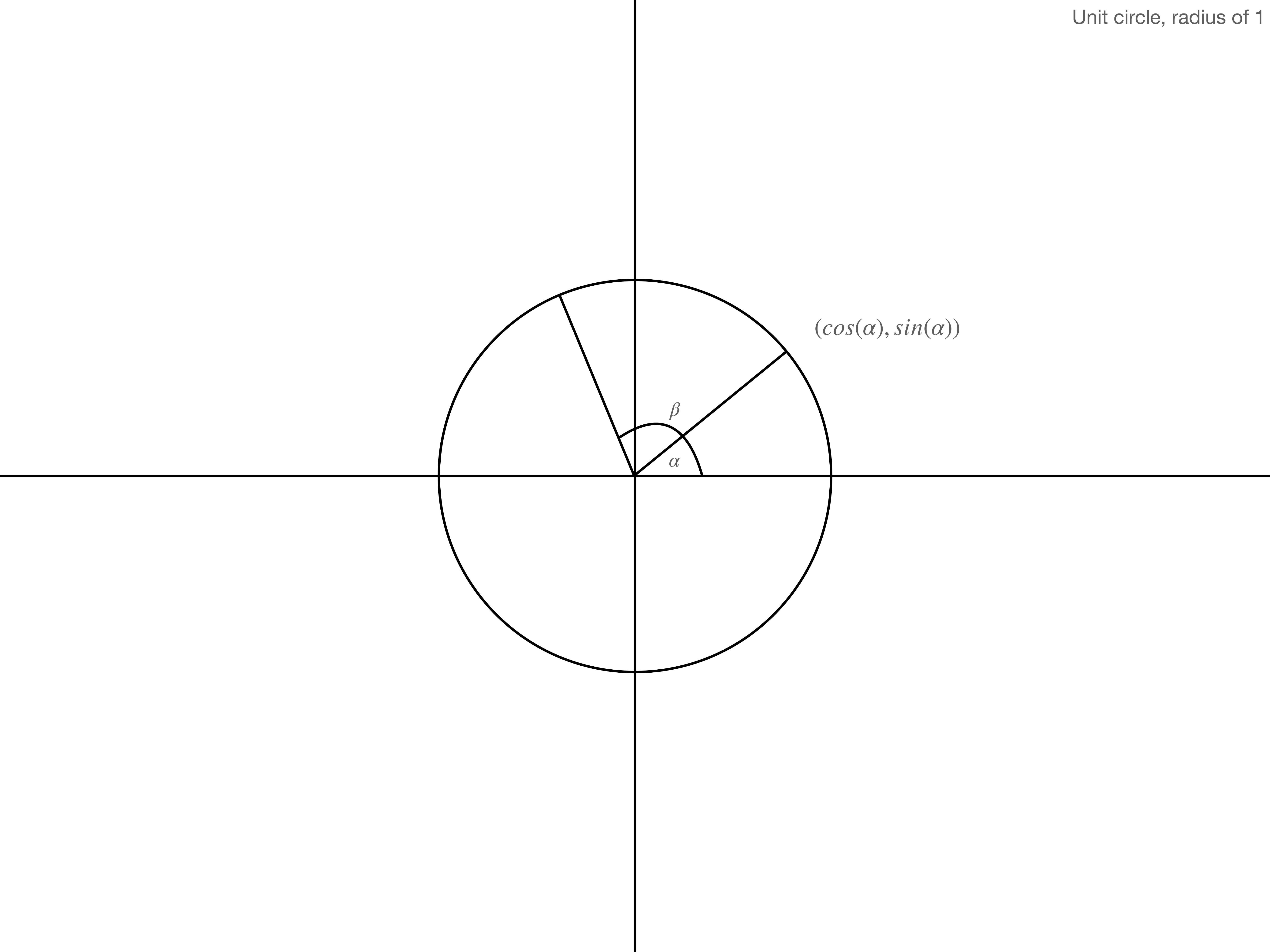
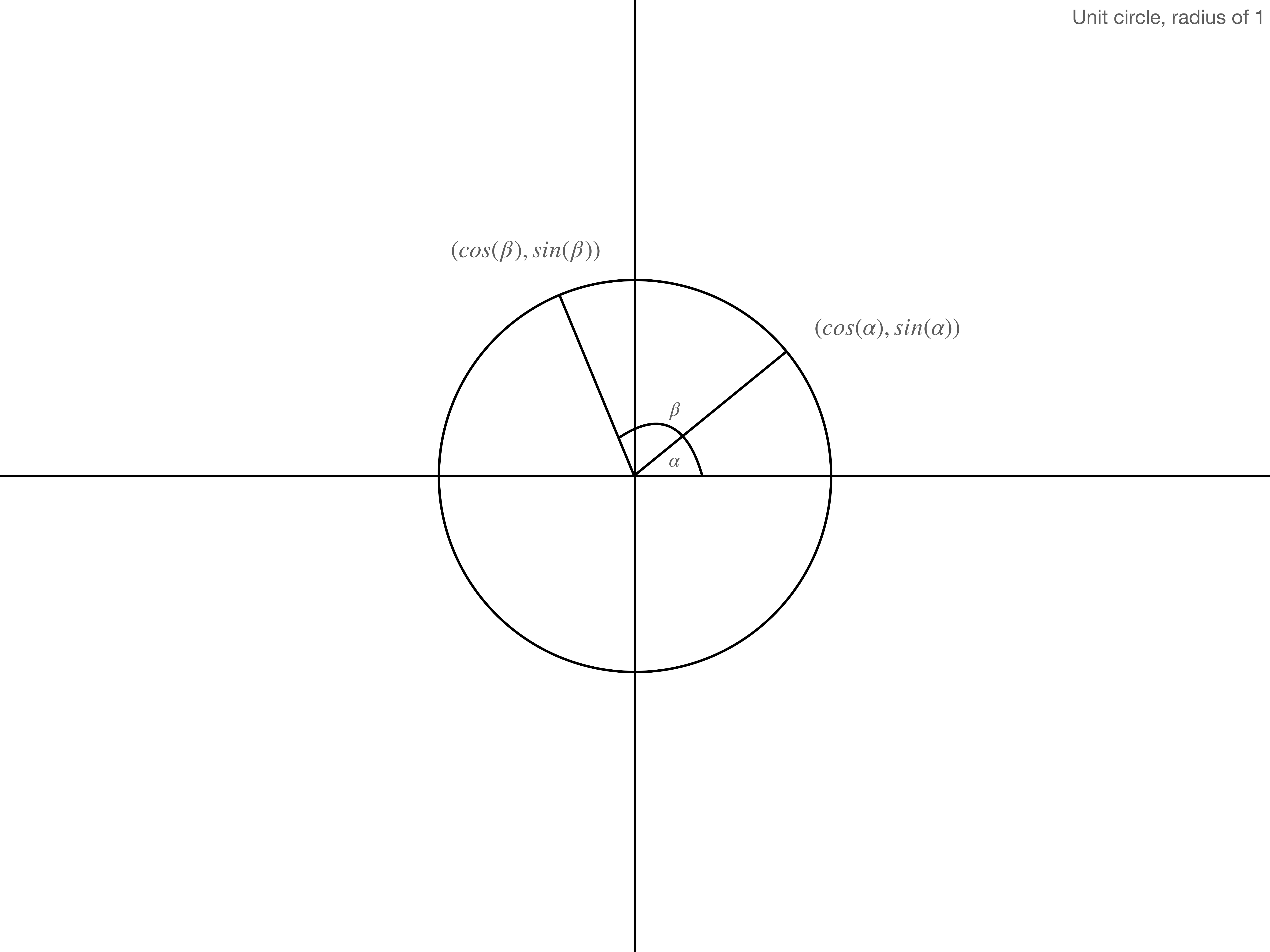
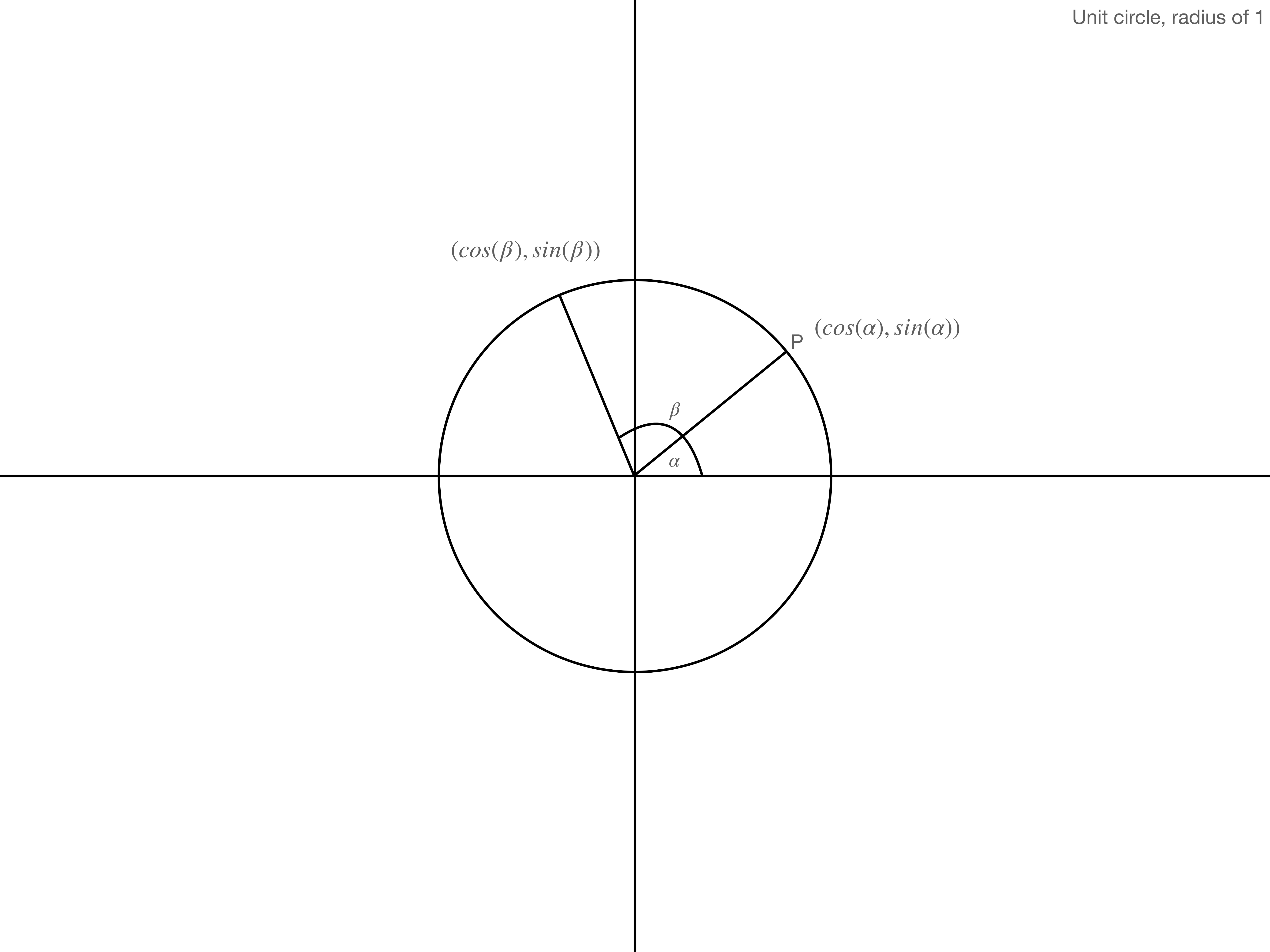


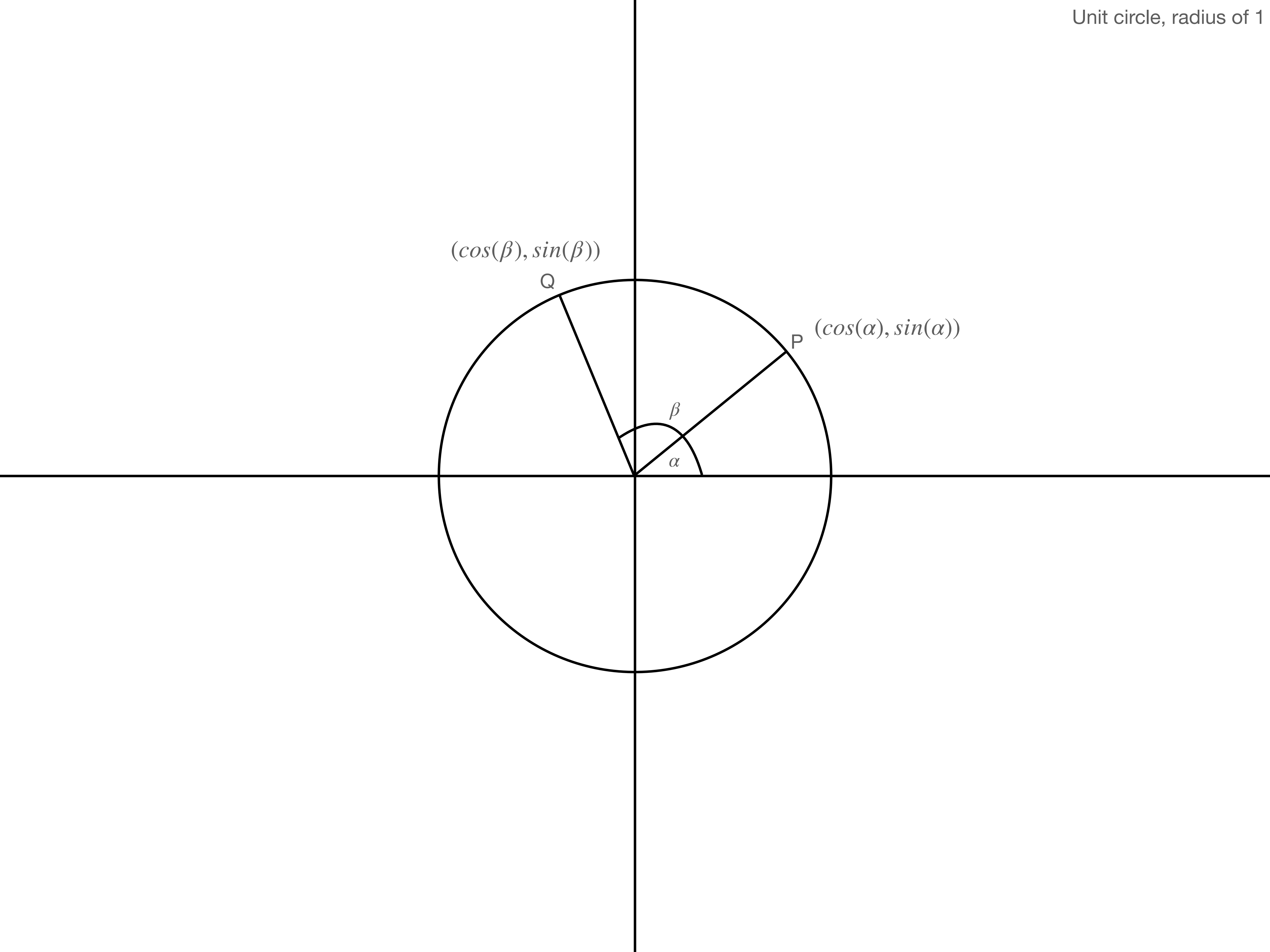

Unit circle, radius of 1
$(cos(\alpha), sin(\alpha))$
β
α

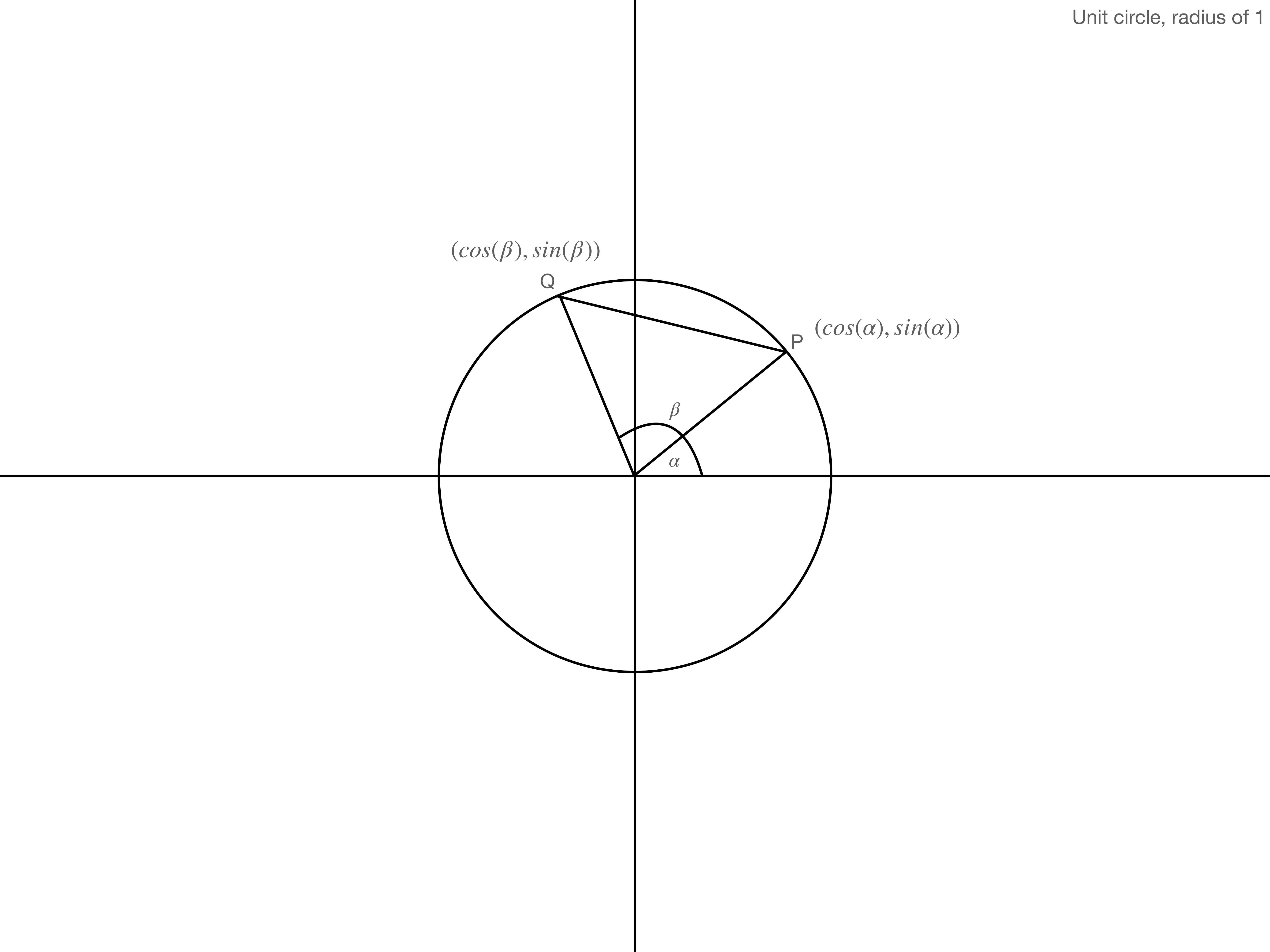

Unit circle, radius of 1
$(cos(\beta), sin(\beta))$
$(cos(\alpha), sin(\alpha))$
β
α



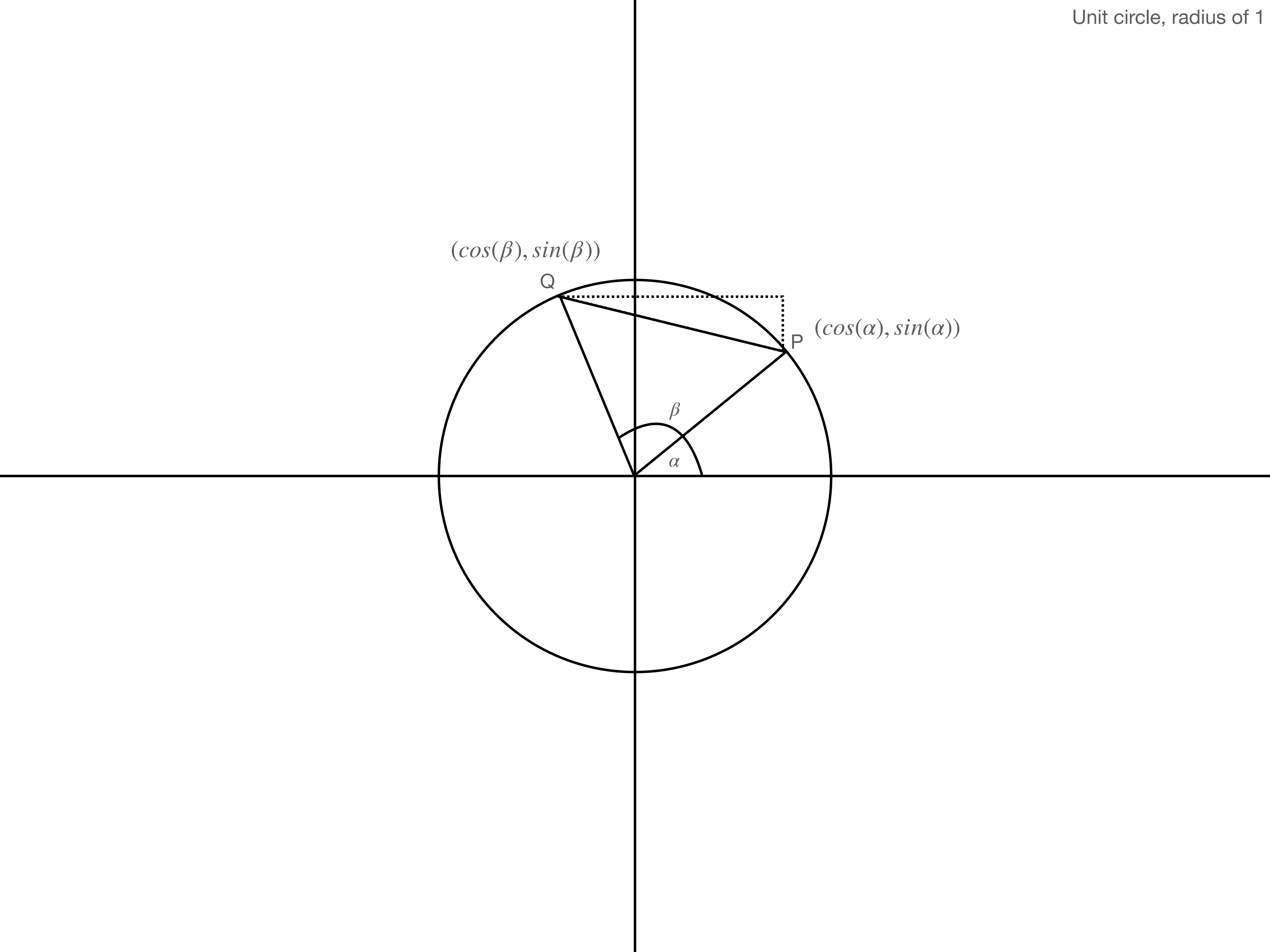
Unit circle, radius of 1
$(cos(\beta), sin(\beta))$
P
$(cos(\alpha), sin(\alpha))$
β
α

Unit circle, radius of 1



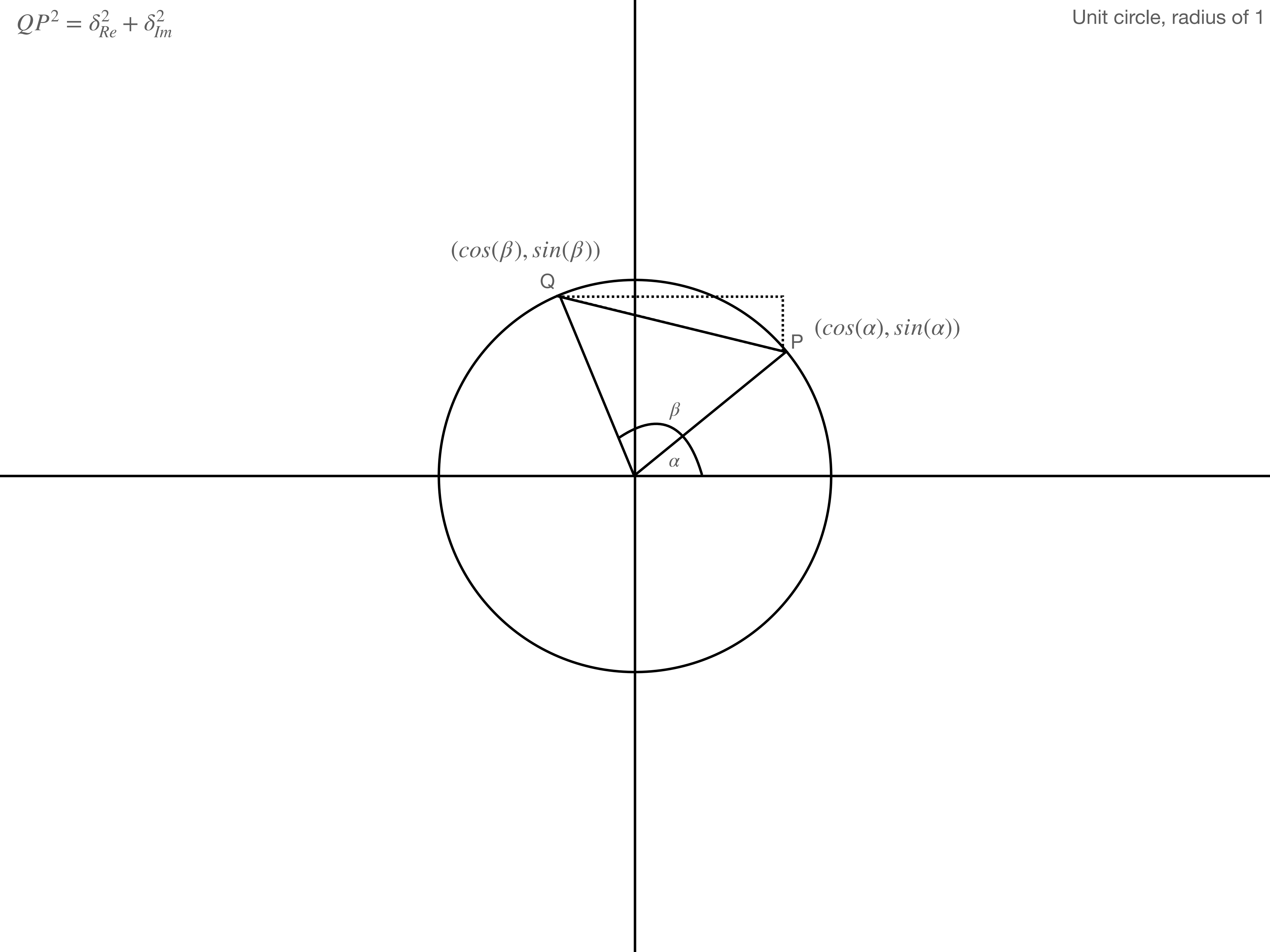


$(cos(\beta), sin(\beta))$
Q
P
$(cos(\alpha), sin(\alpha))$
β
α


Unit circle, radius of 1
(cos(β), sin(β))
Q
P
(cos(α), sin(α))
β
α

$QP^2 = \delta_{Re}^2 + \delta_{Im}^2$

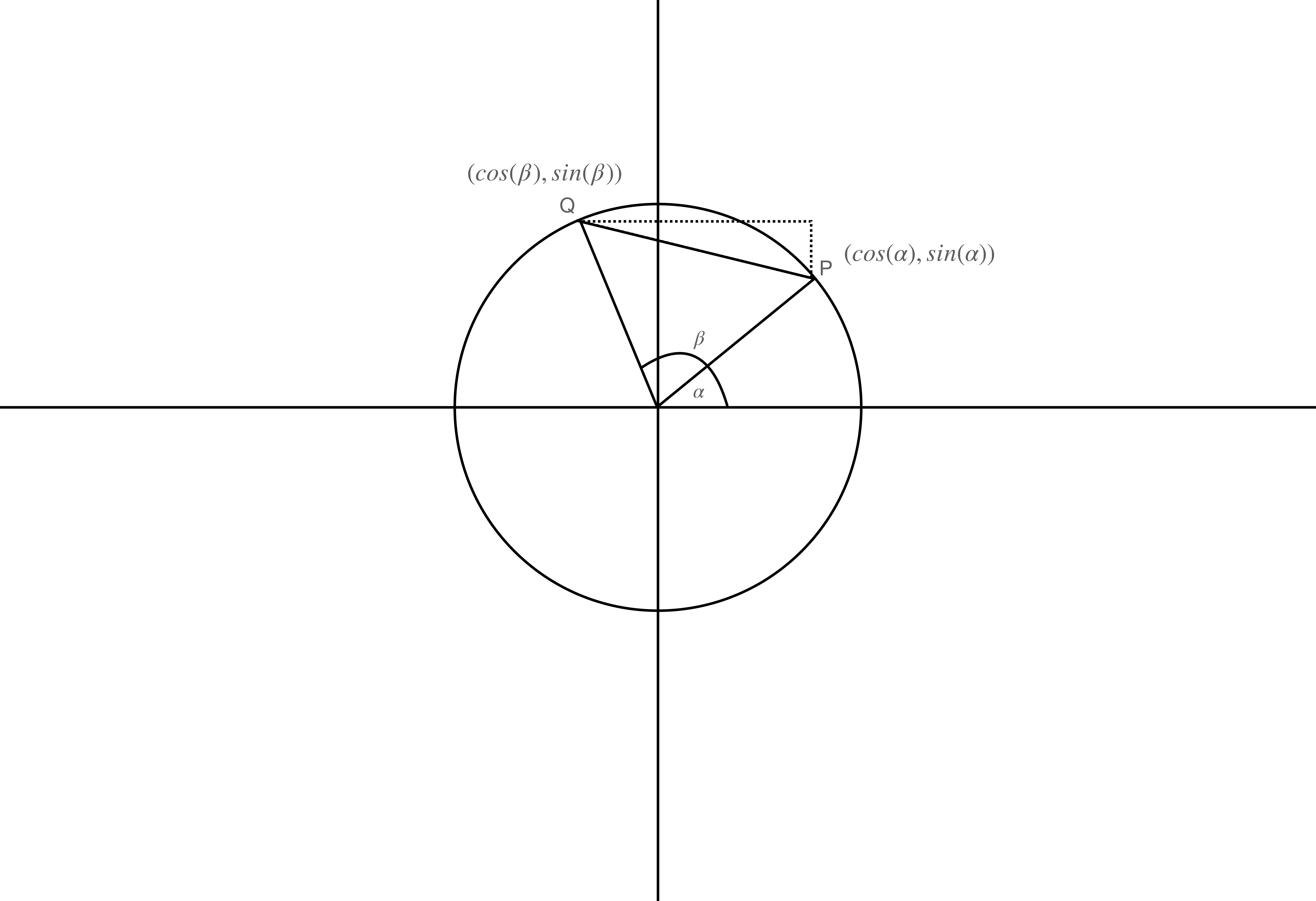
Unit circle, radius of 1



$$QP^2 = \delta_{Re}^2 + \delta_{Im}^2$$

$$QP^2 = (cos(\beta) - cos(\alpha))^2 + (sin(\beta) - sin(\alpha))^2$$

Unit circle, radius of 1



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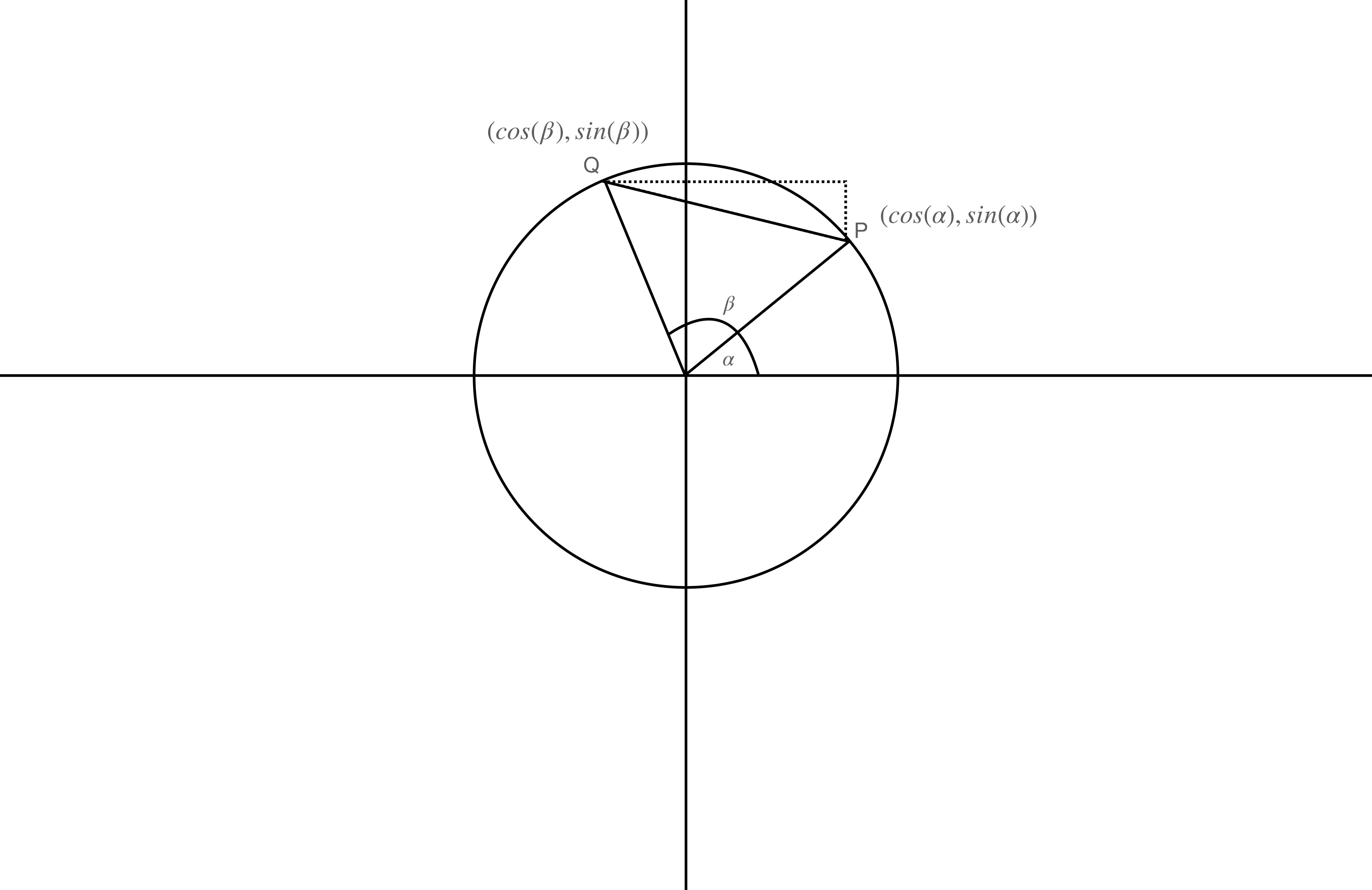
$$QP^2 = \left(cos^2(\beta) - 2cos(\beta)cos(\alpha) + cos^2(\alpha)\right) + \left(sin^2(\beta) - 2sin(\beta)sin(\alpha) + sin^2(\alpha)\right)$$



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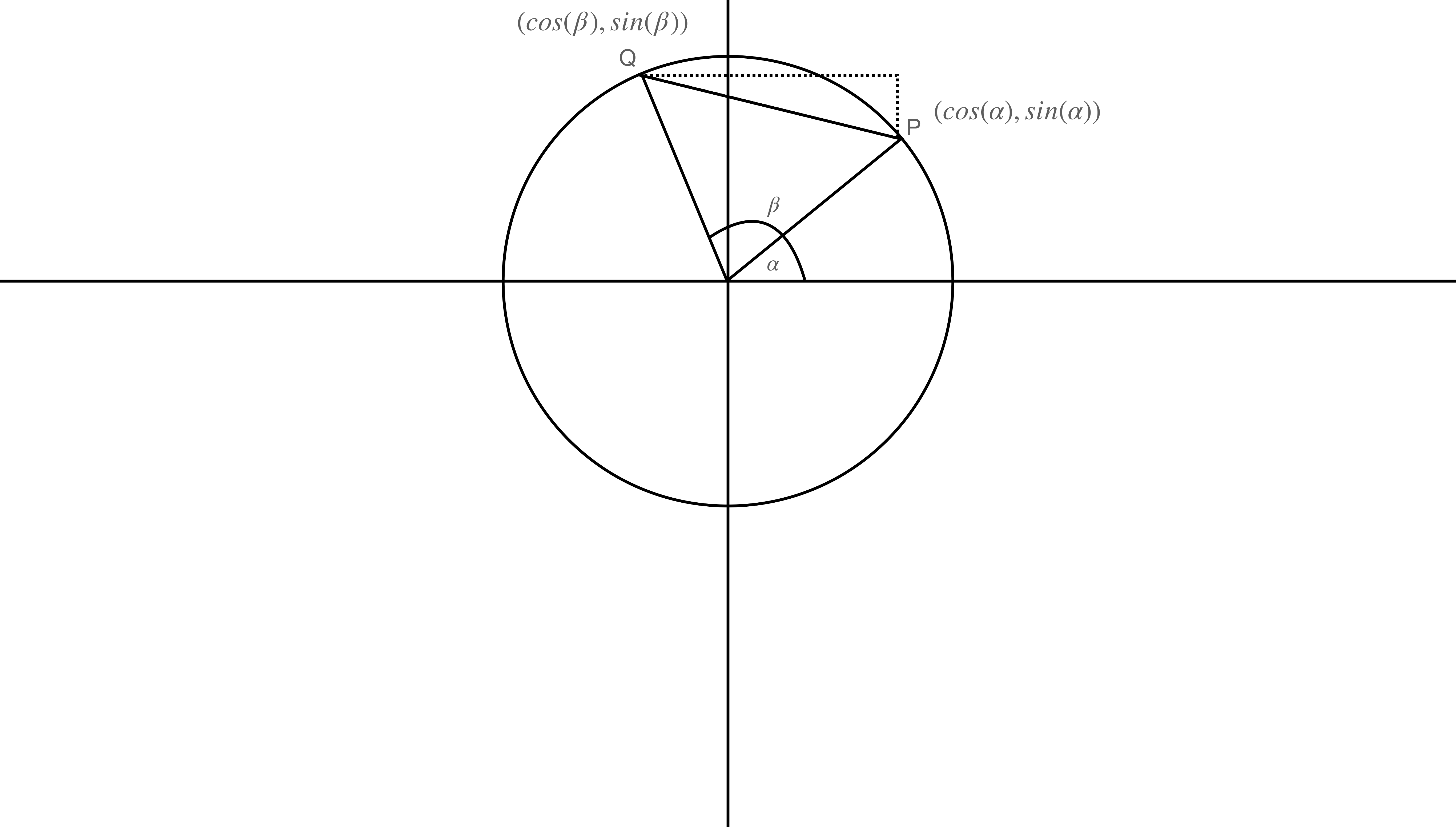

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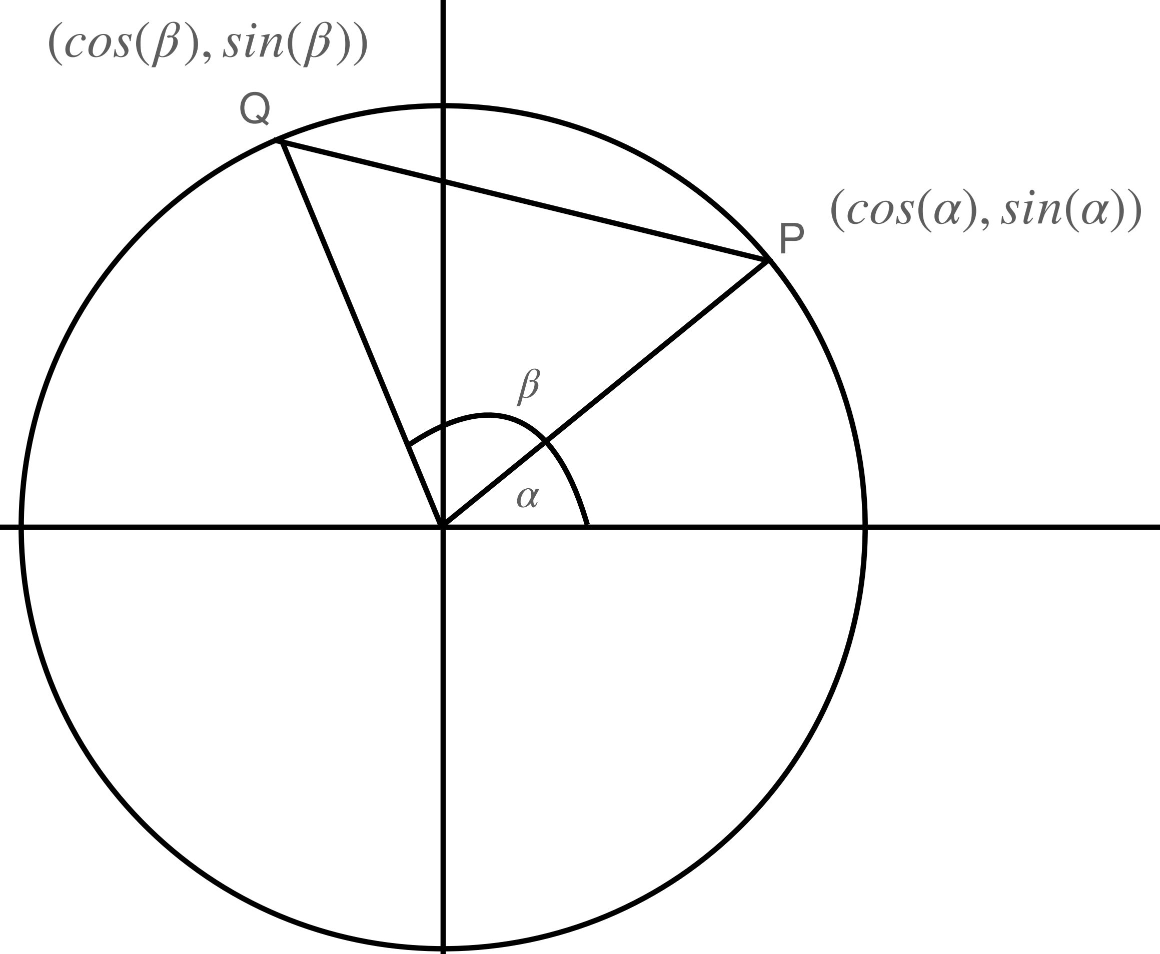

NOW ROTATE BOTH LINES BY ANGLE -alpha

(i.e., add -alpha to each angle)

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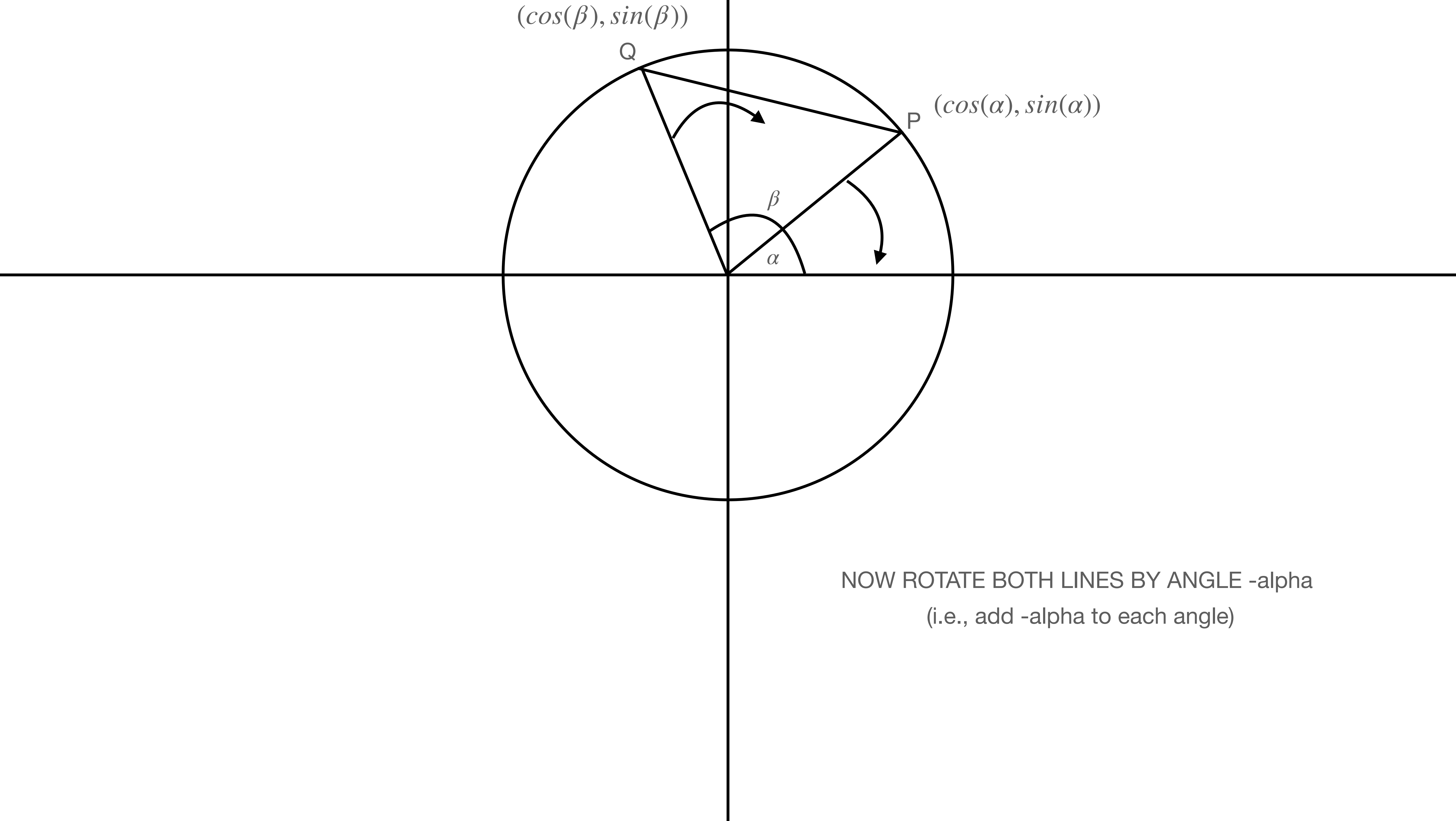

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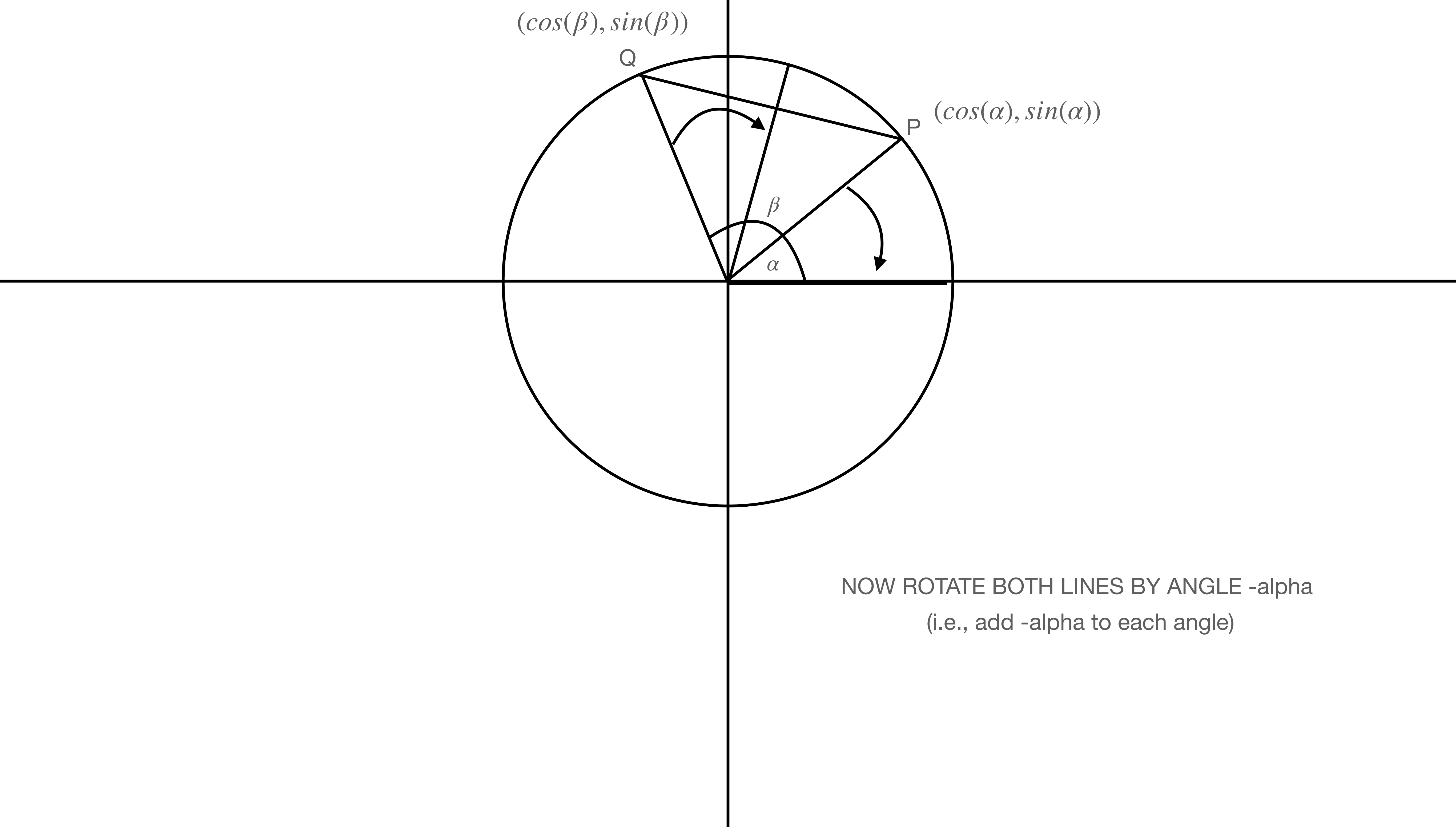

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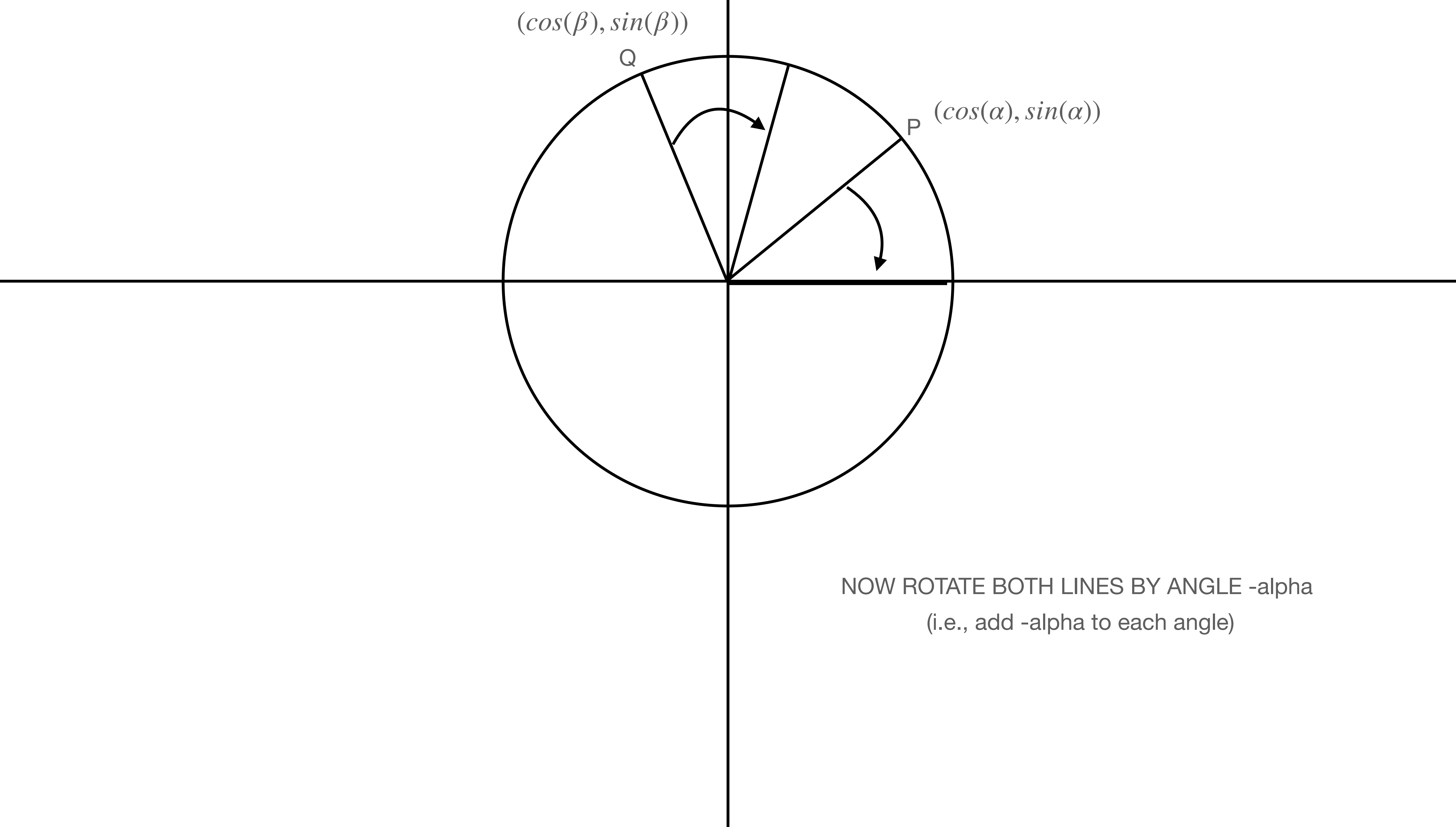

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(i.e., add -alpha to each angle)

Unit circle, radius of 1

$$QP^2 = \delta_{Re}^2 + \delta_{Im}^2$$

$$QP^2 = (cos(\beta) - cos(\alpha))^2 + (sin(\beta) - sin(\alpha))^2$$

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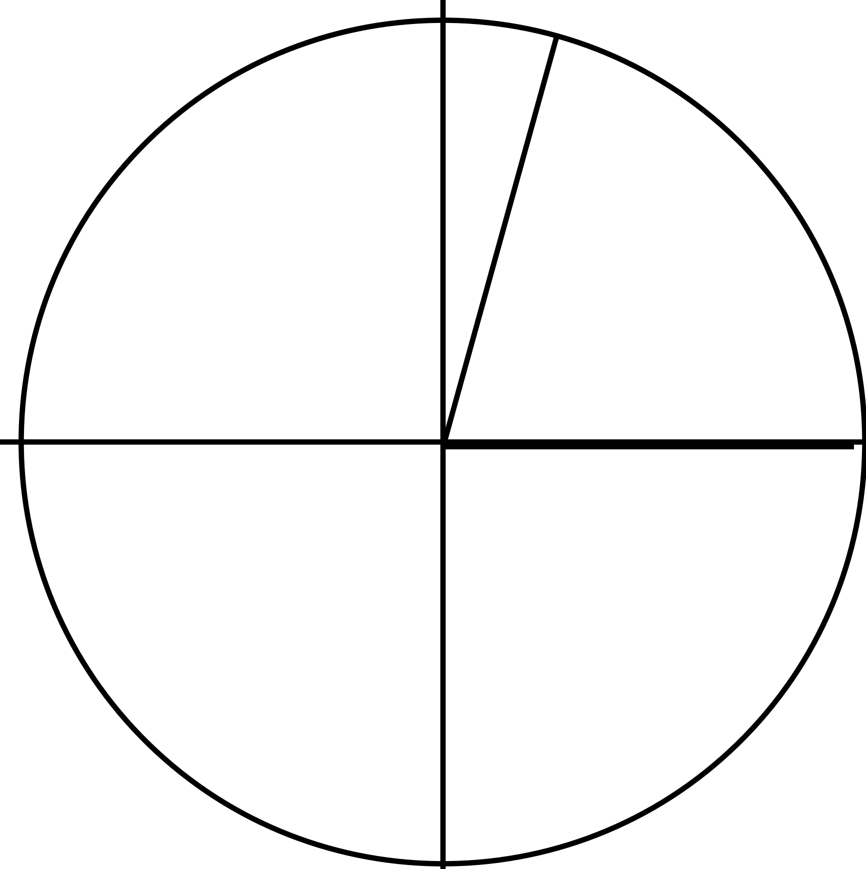

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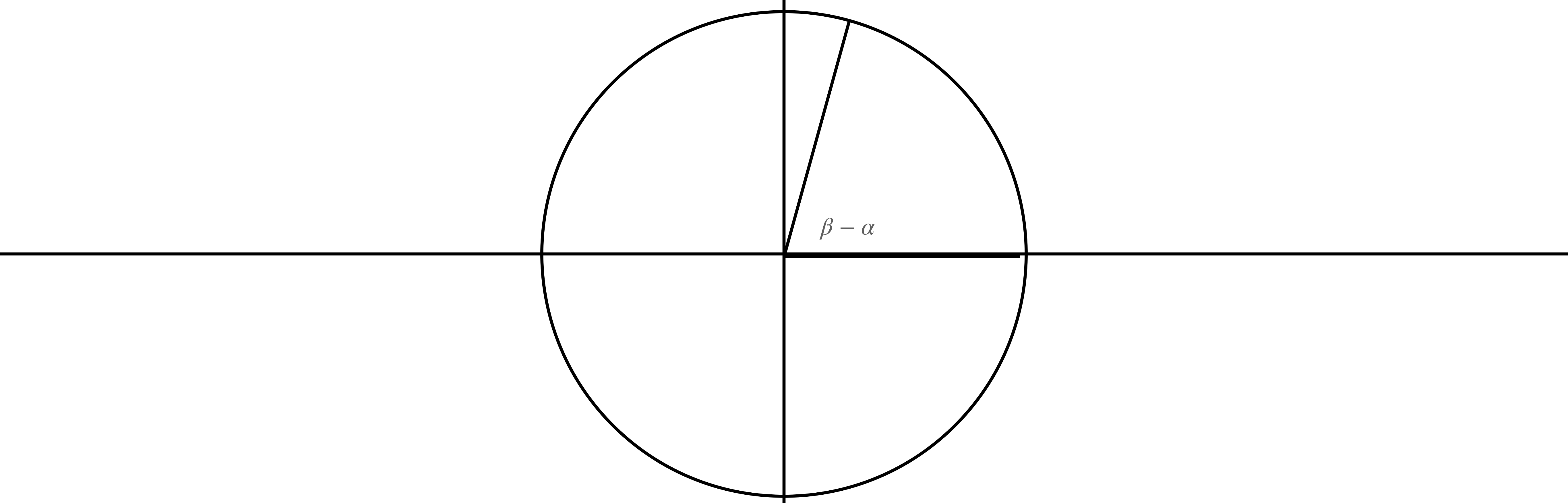

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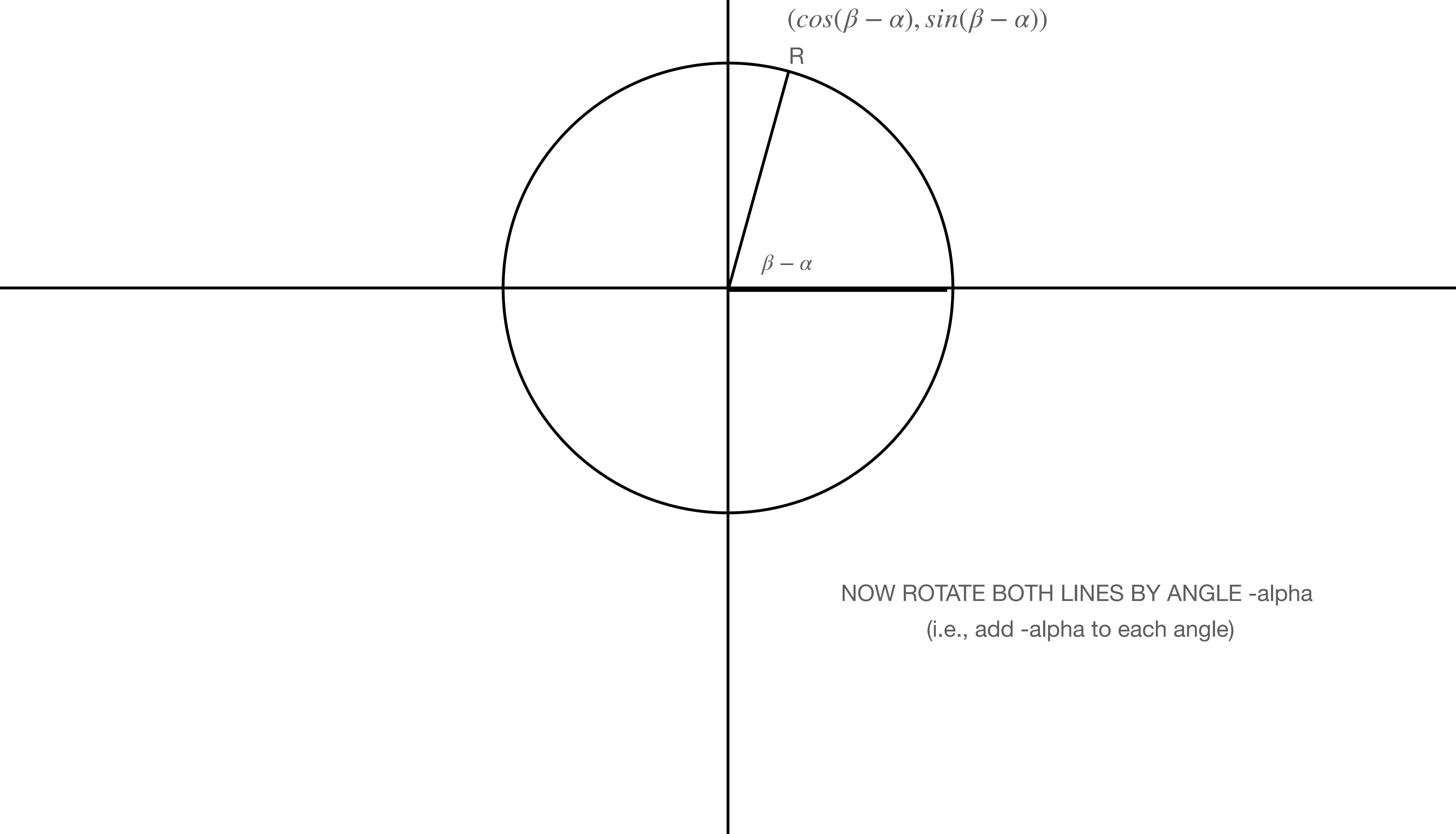

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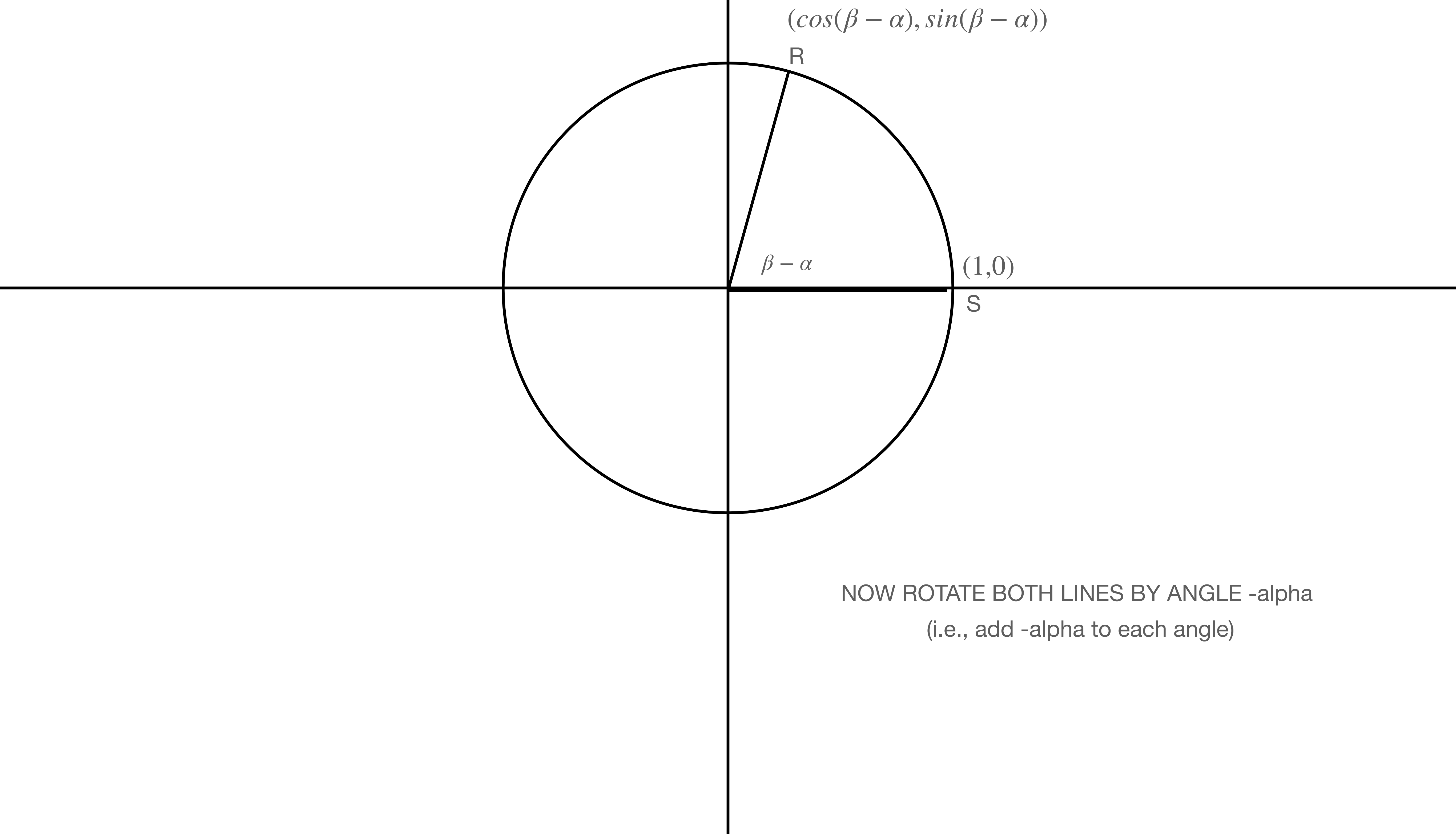

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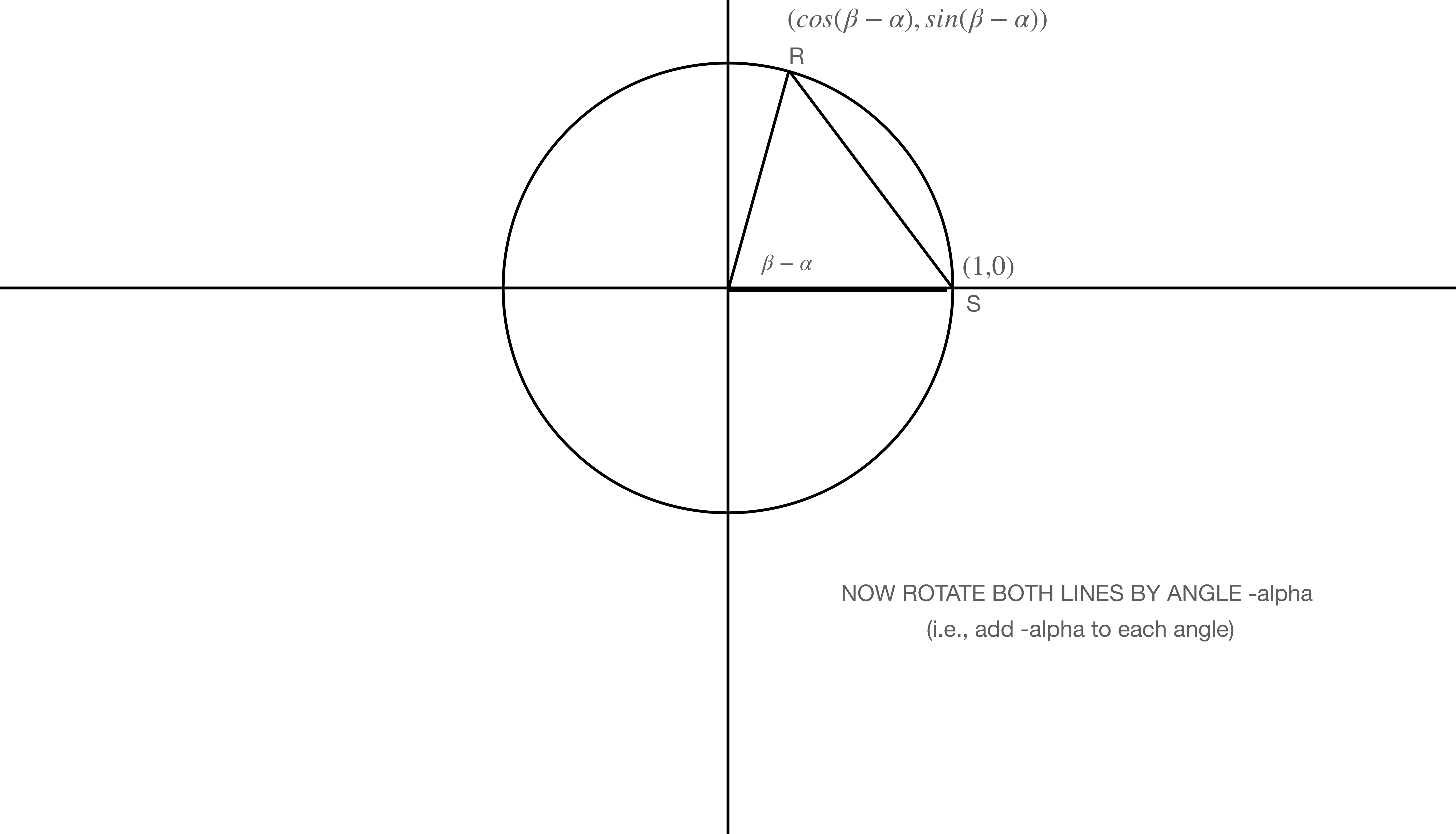

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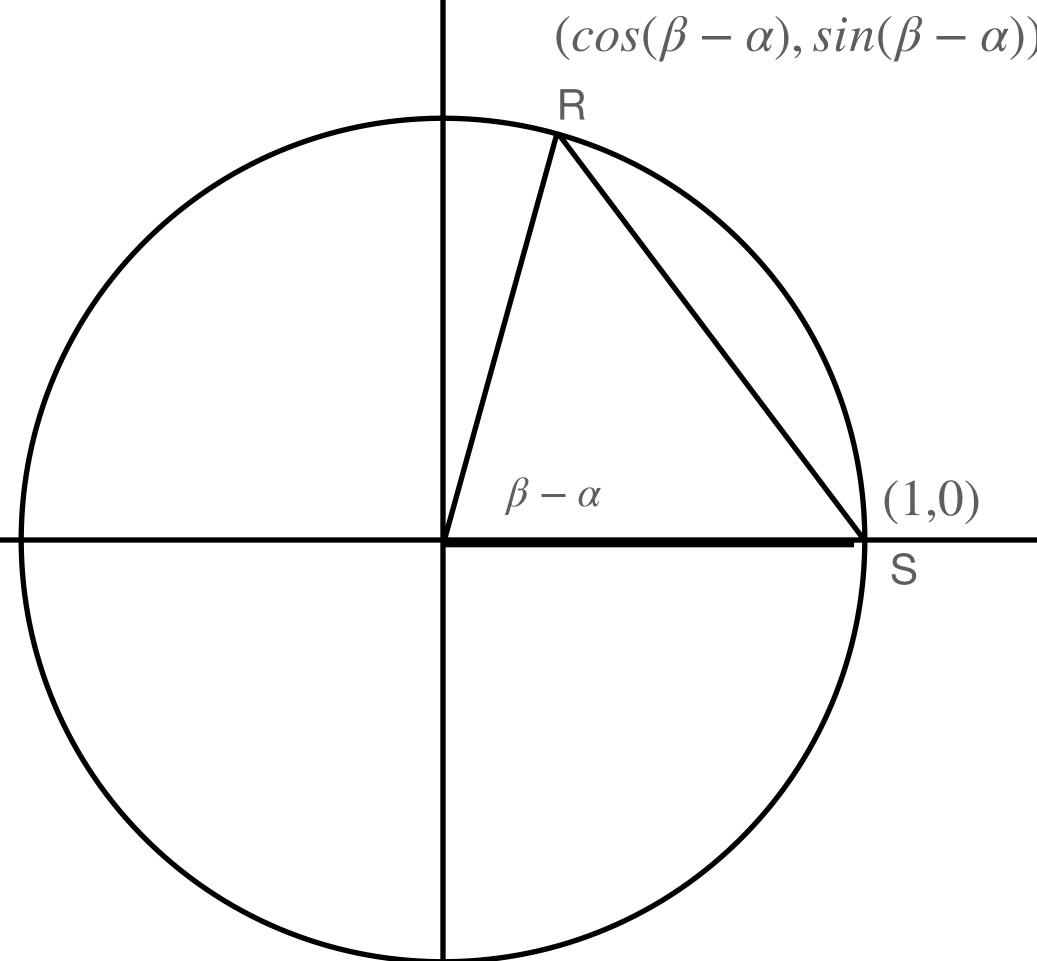

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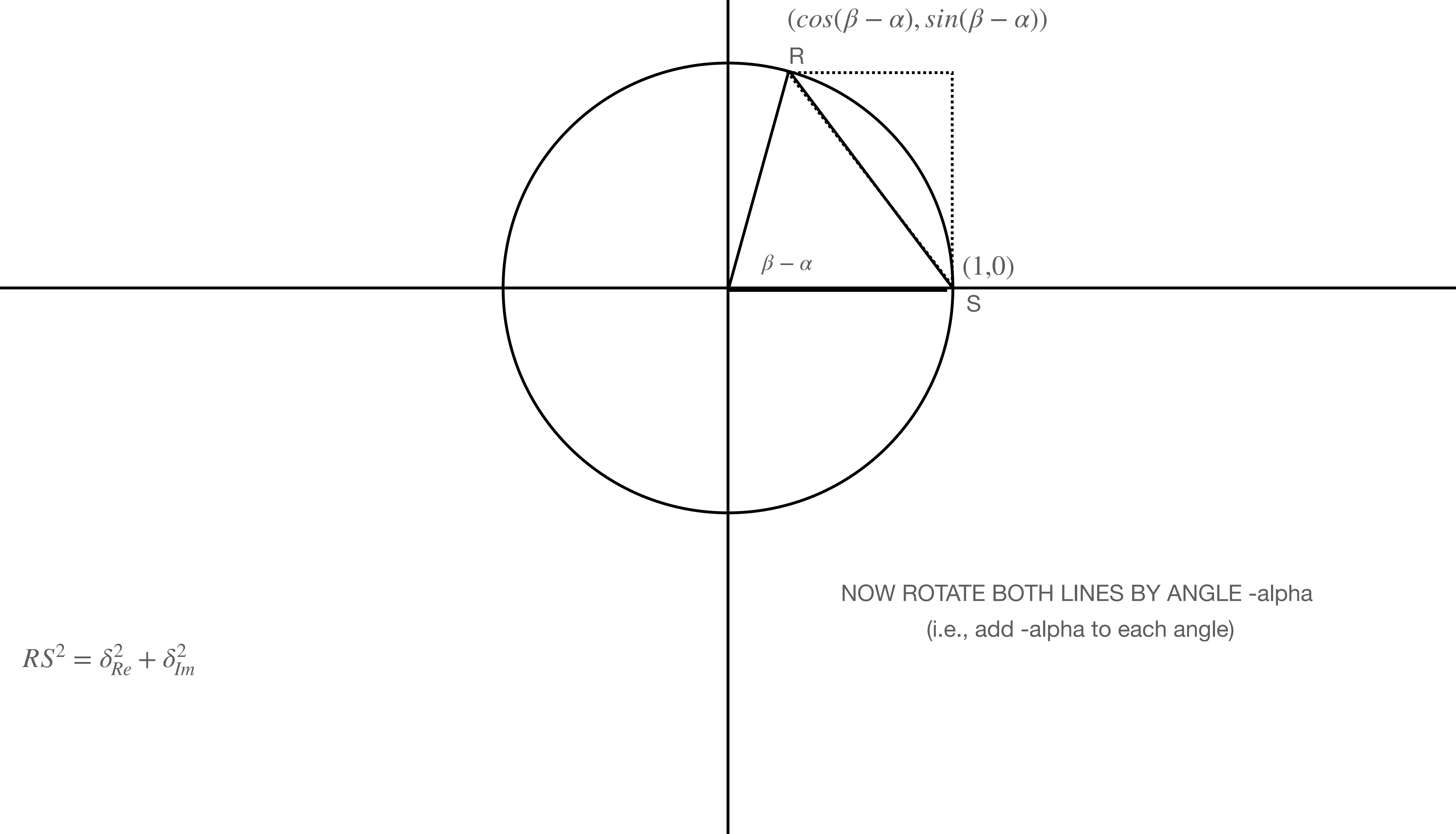

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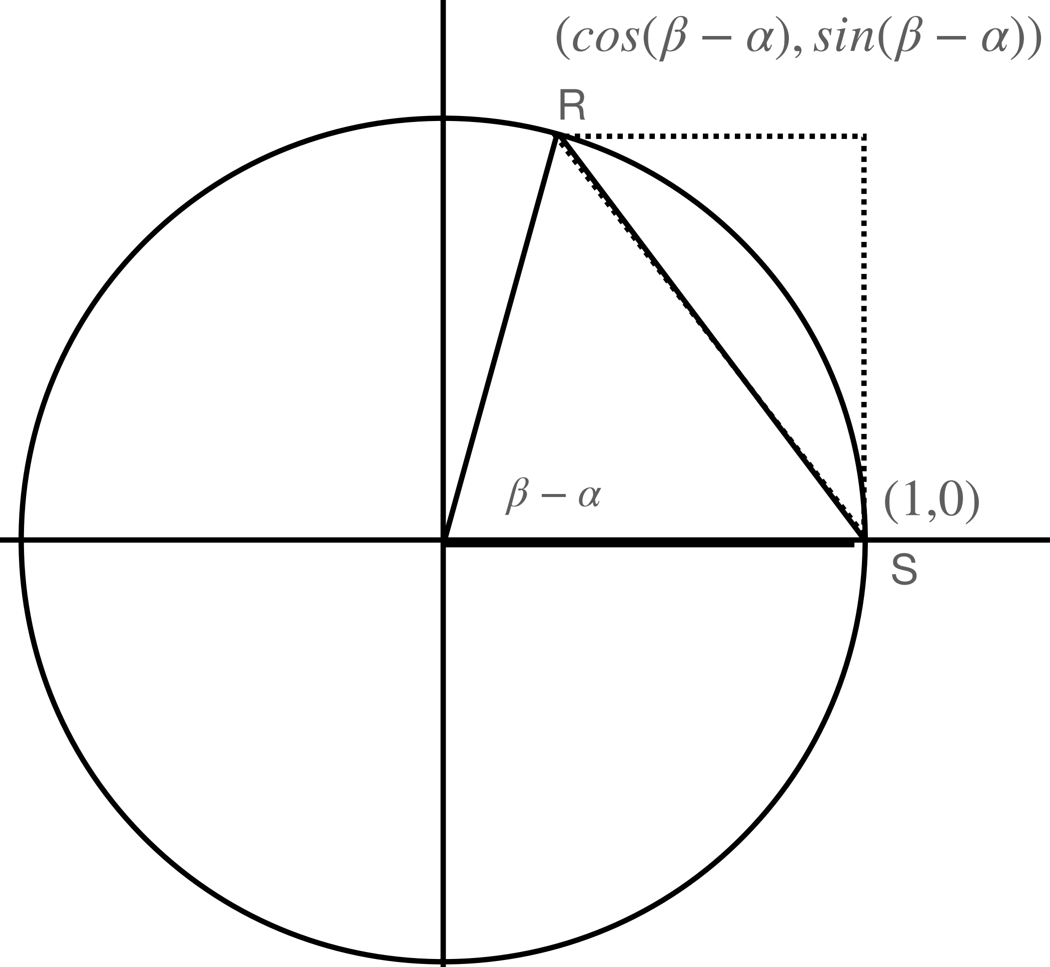

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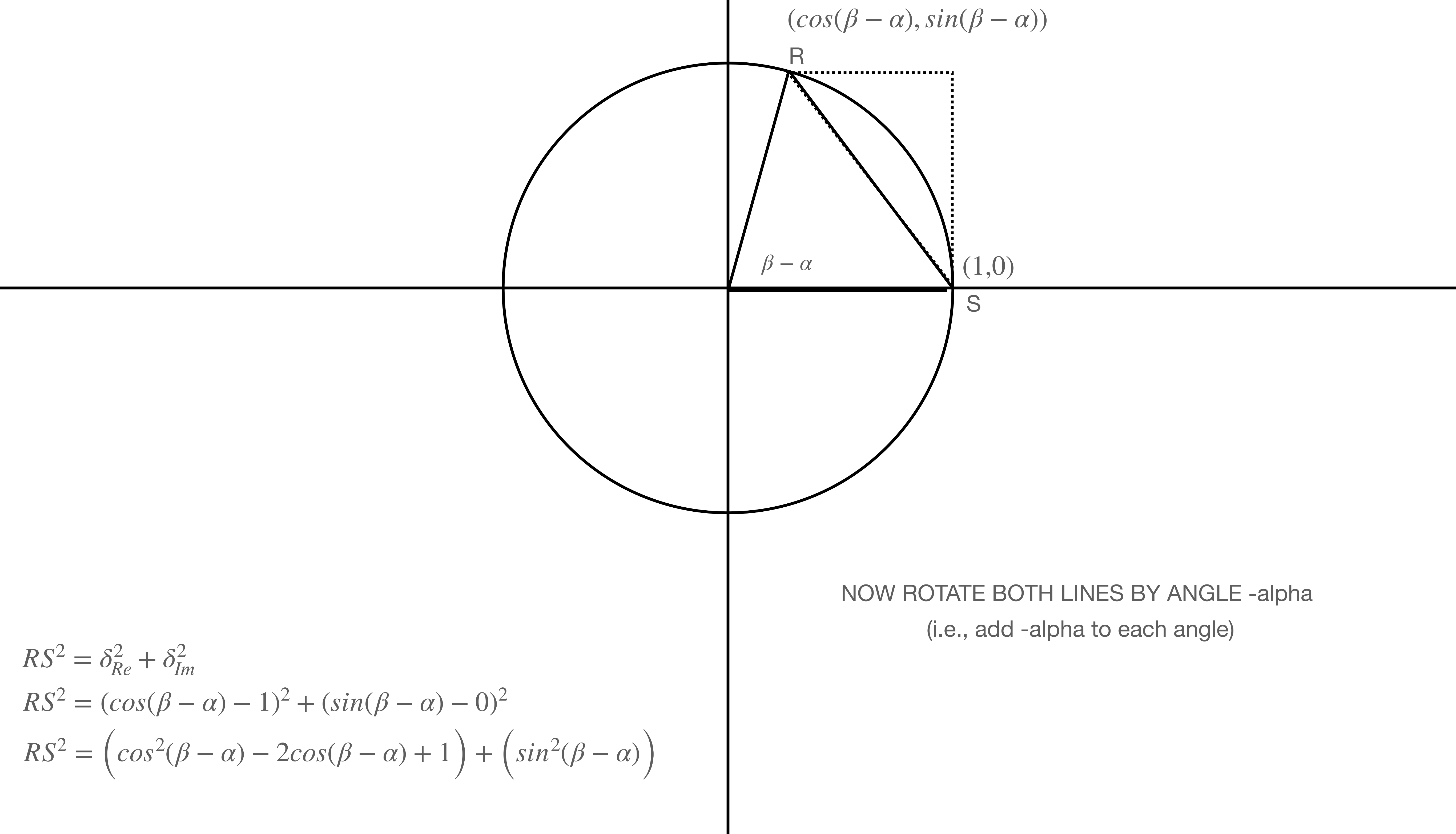

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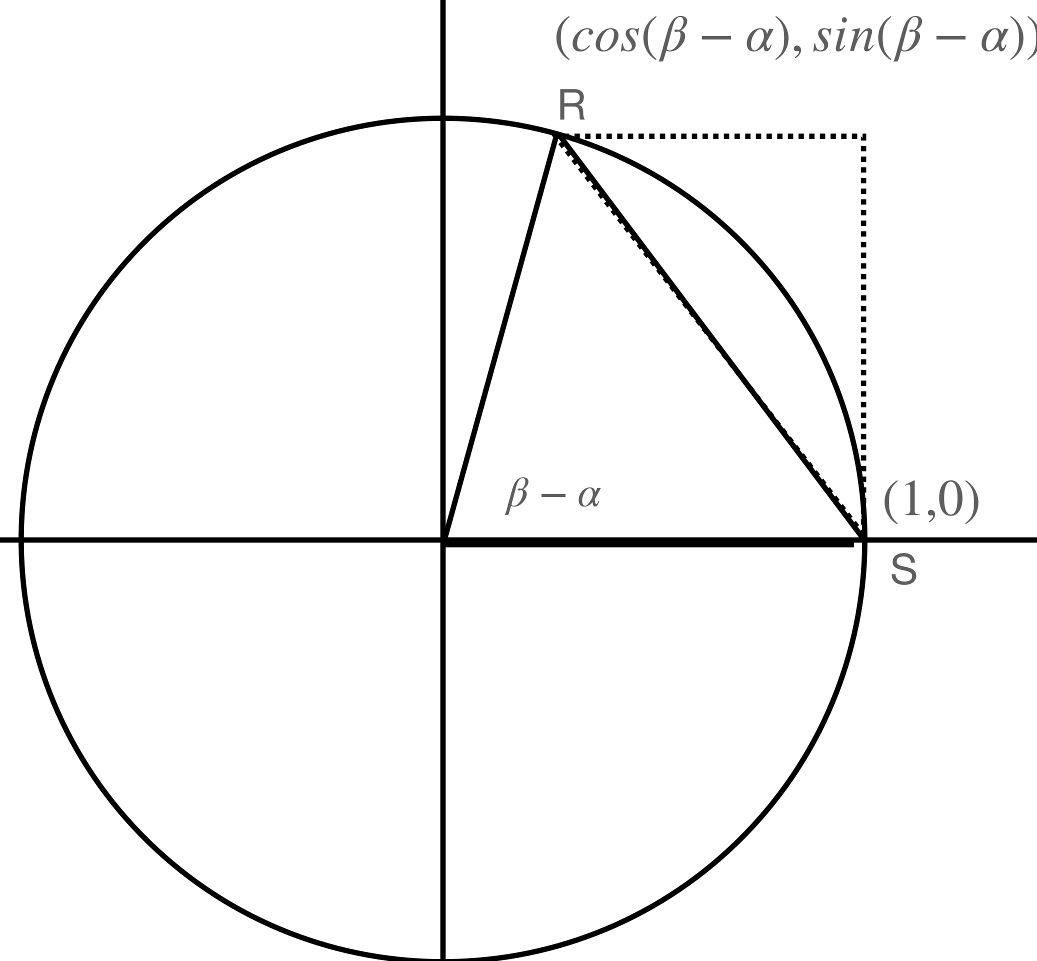

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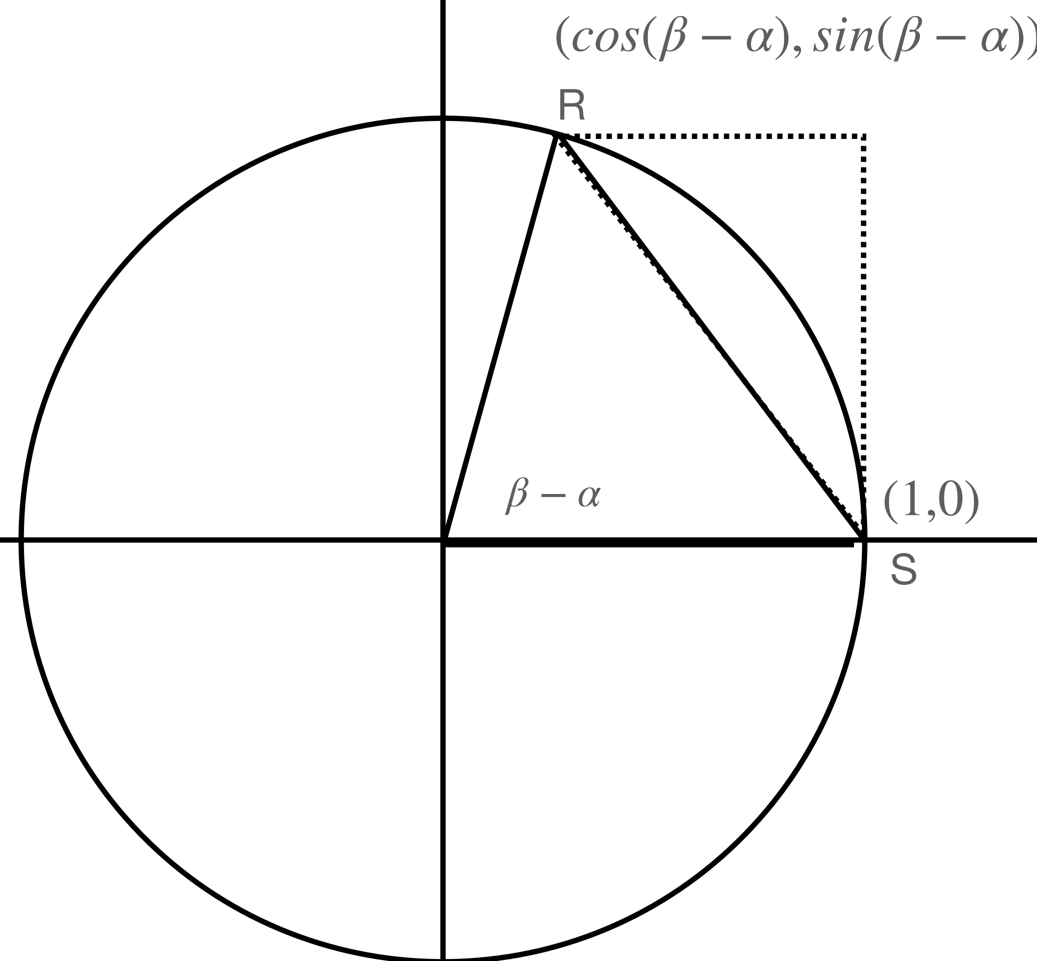

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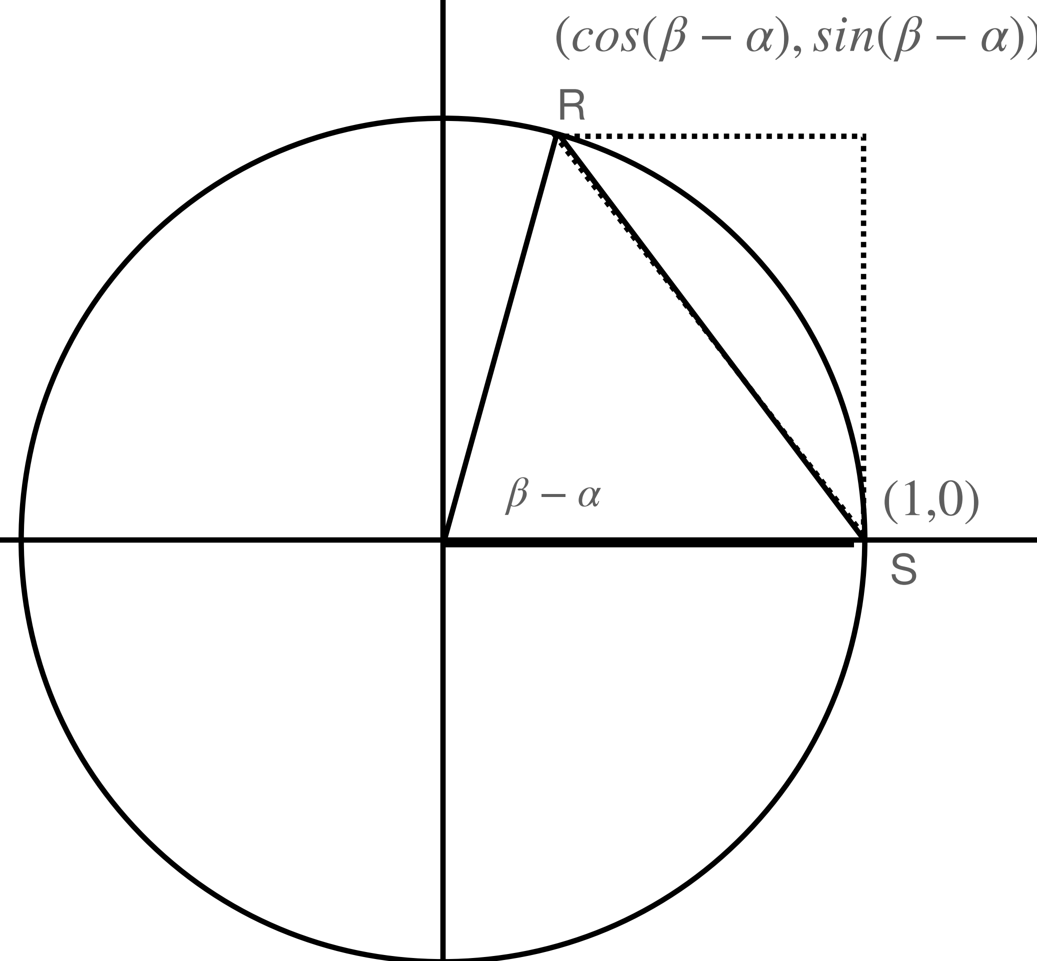

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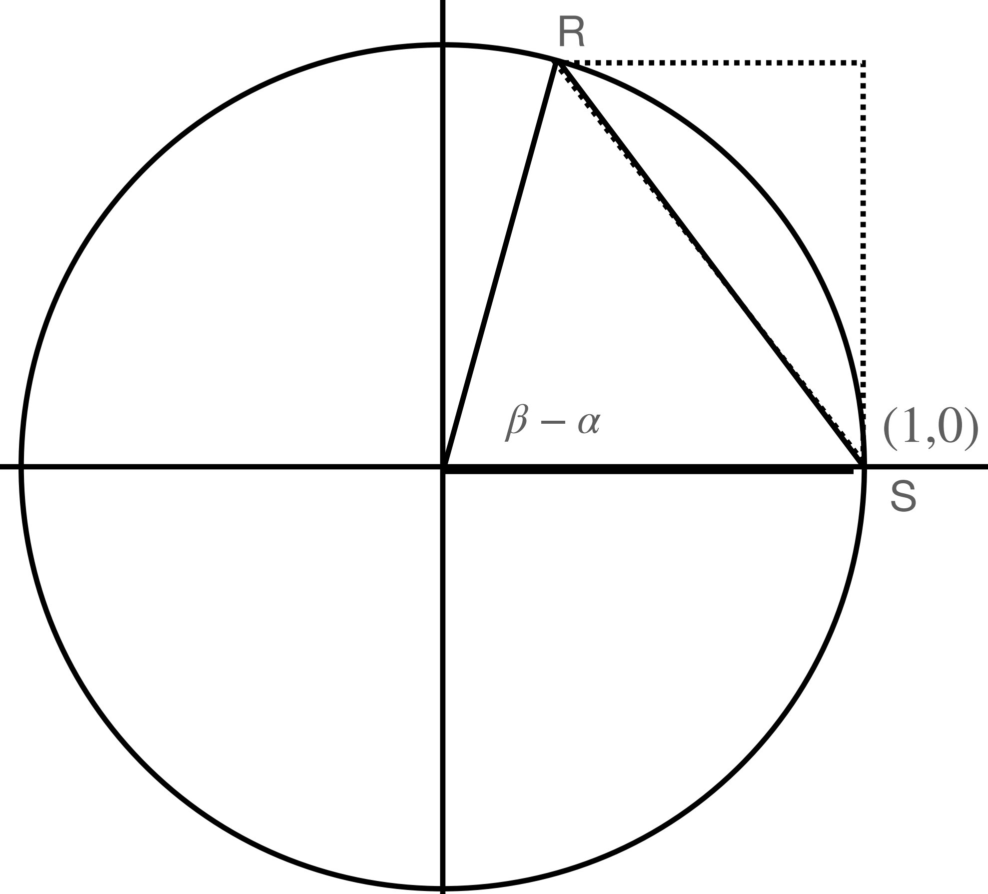
$(cos(\beta - \alpha), sin(\beta - \alpha))$



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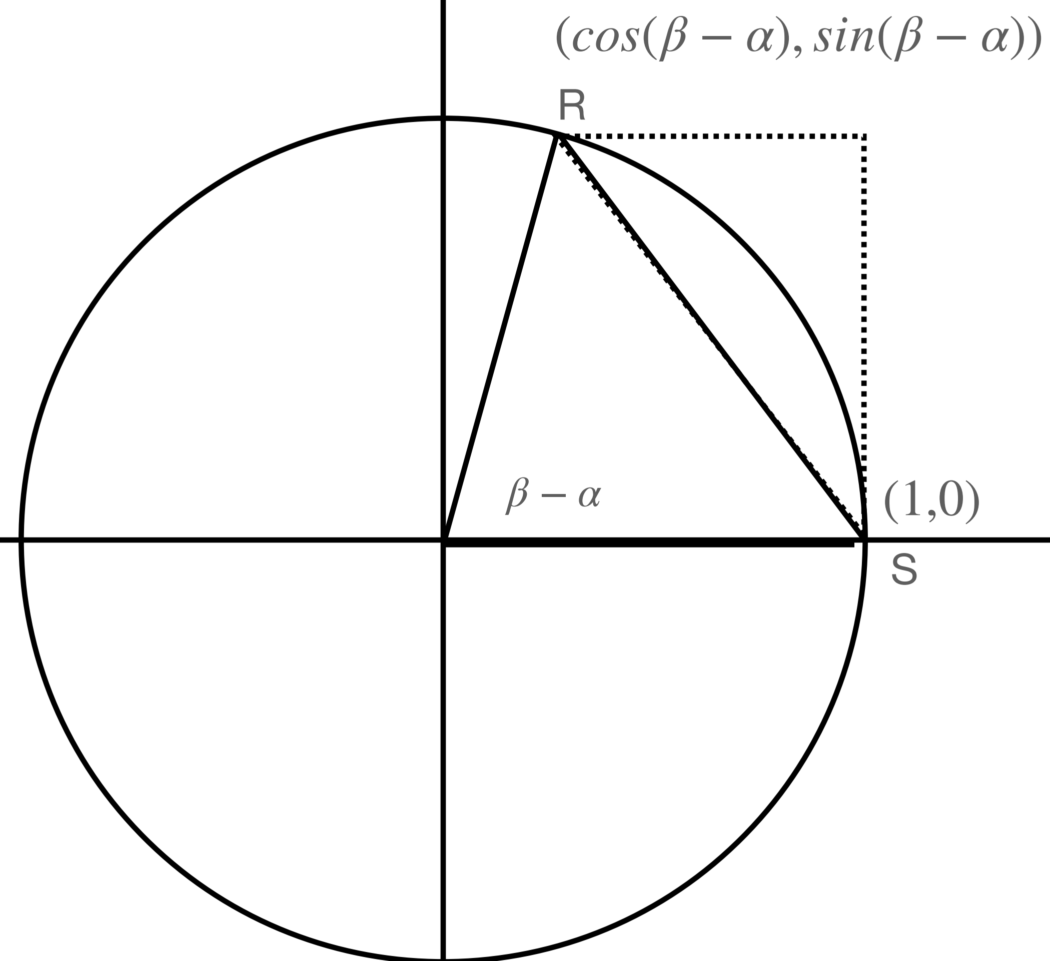

THIS TIME WE PLACED NO LIMITS ON ANGLES INVOLVED!

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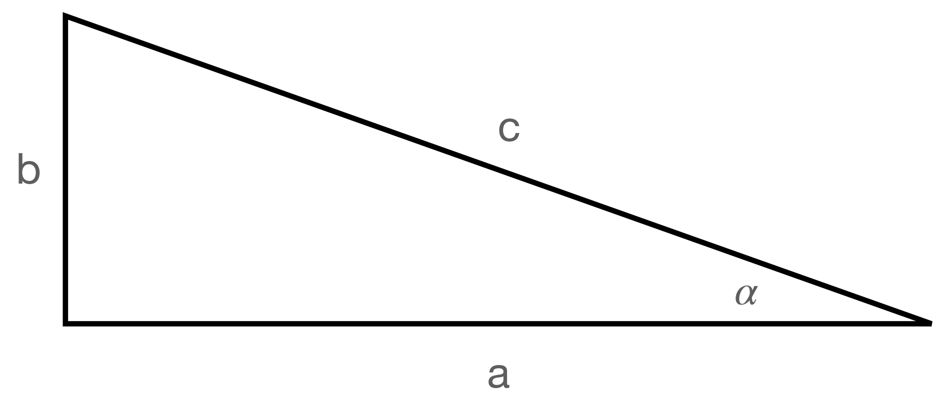
In a right triangle, by definition



$$sin(\alpha) = \frac{opp}{hyp} = \frac{b}{c}$$

$$cos(\alpha) = \frac{adj}{hyp} = \frac{a}{c}$$

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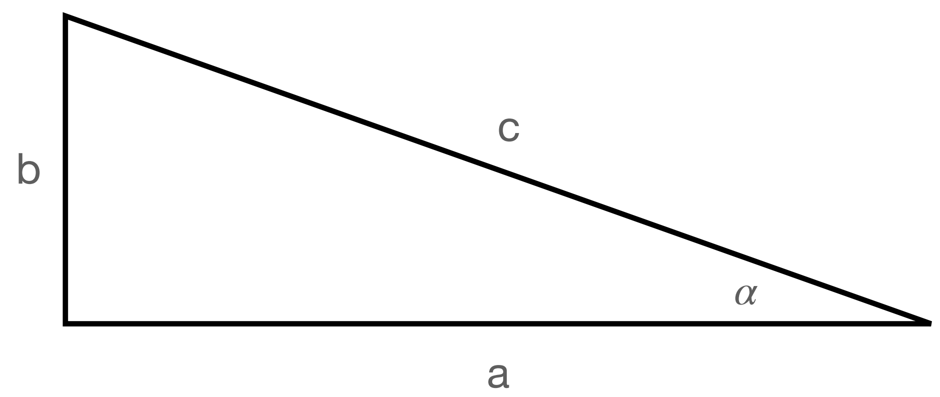

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For complementary angles, their sines and cosines are exchanged

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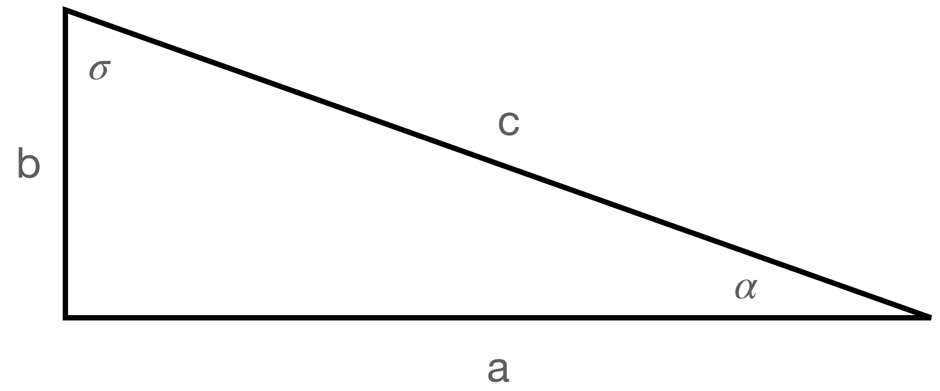

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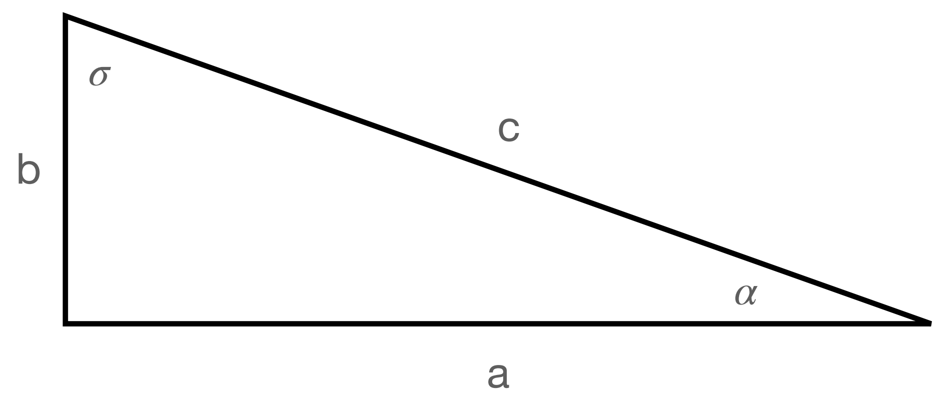

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$$\sigma = \frac{\pi}{2} - \alpha$$

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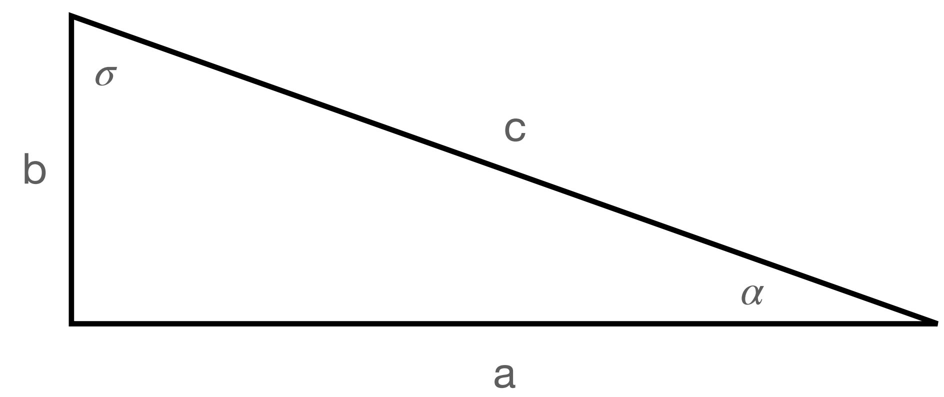

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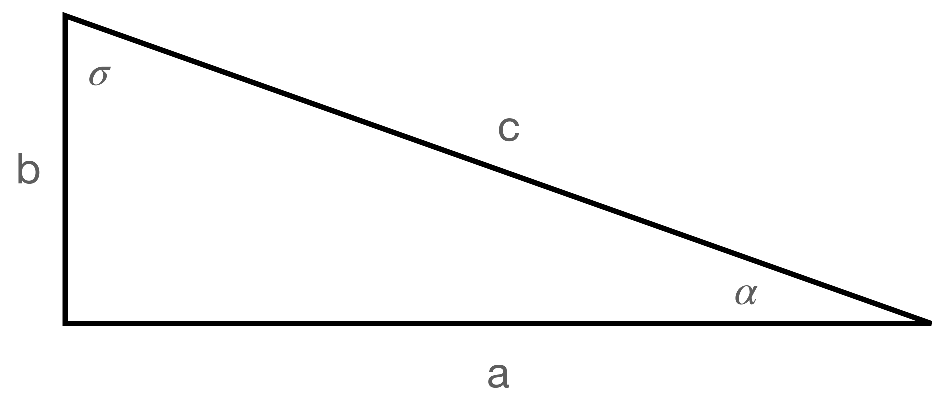

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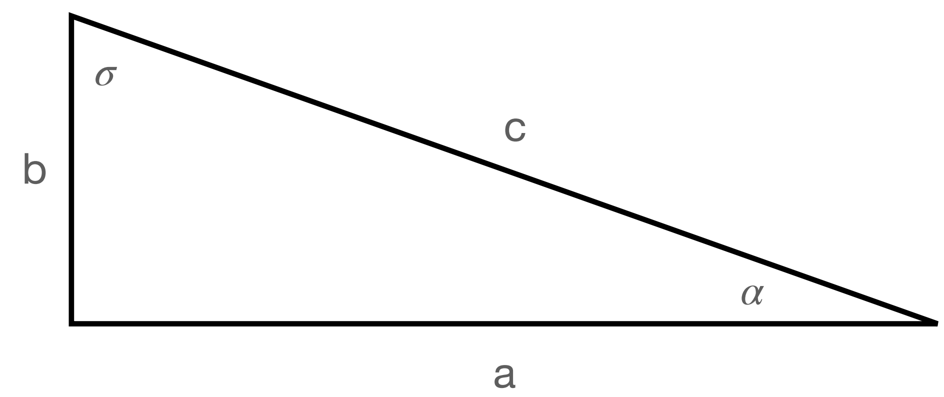

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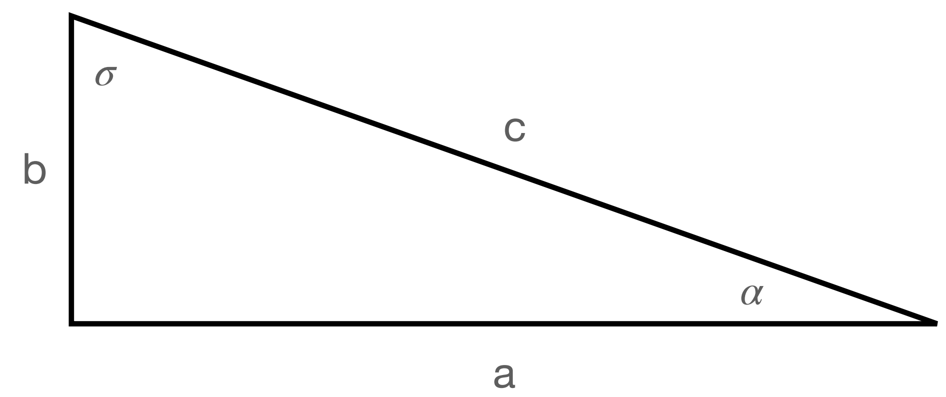
This is a triangle-centric view where we treat all angles as if they were less than 90 degrees

In a right triangle, by definition



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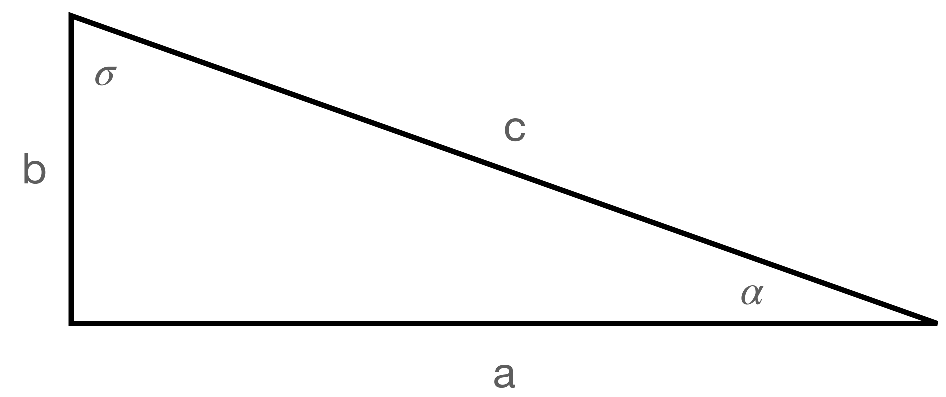

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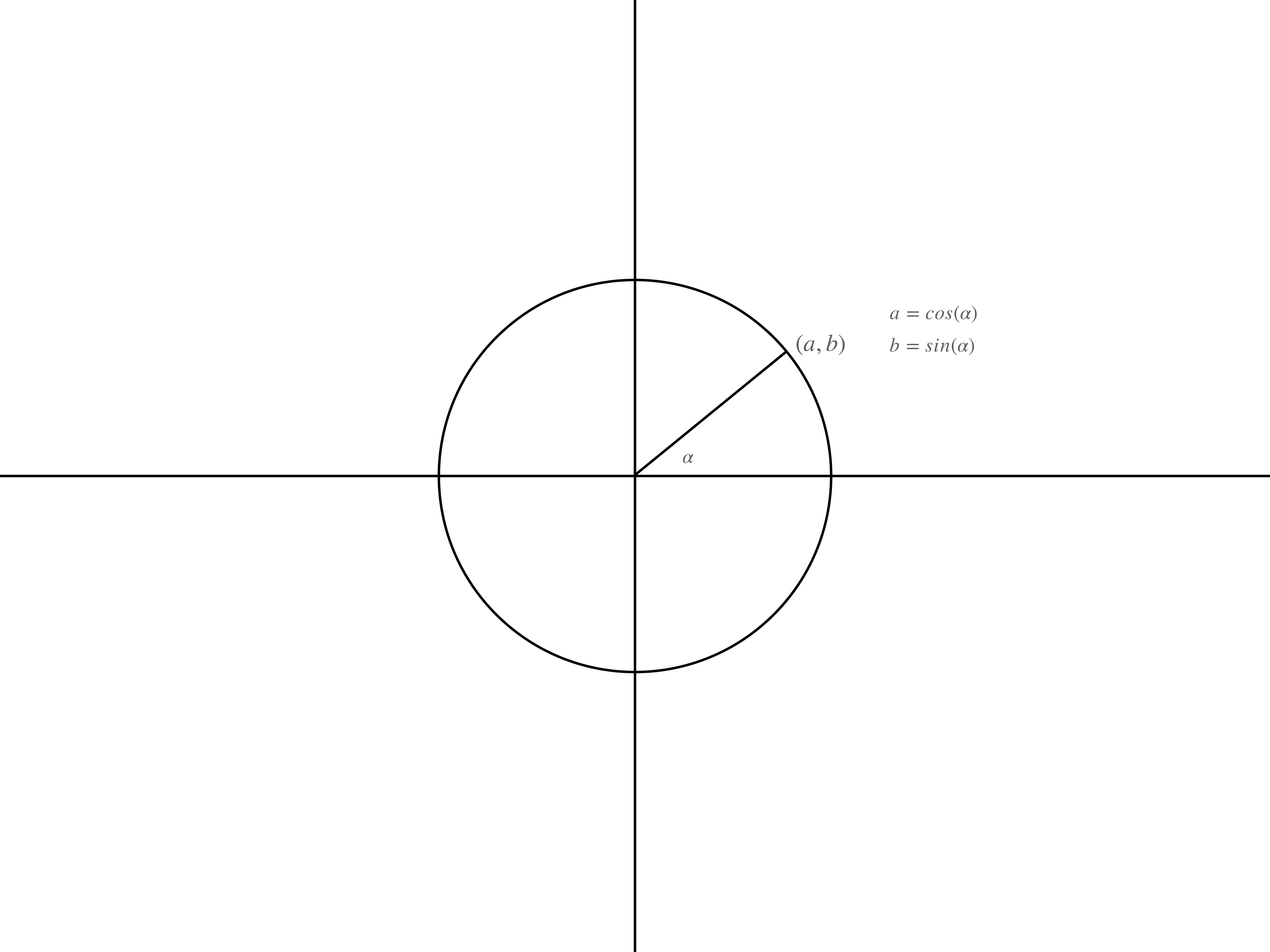
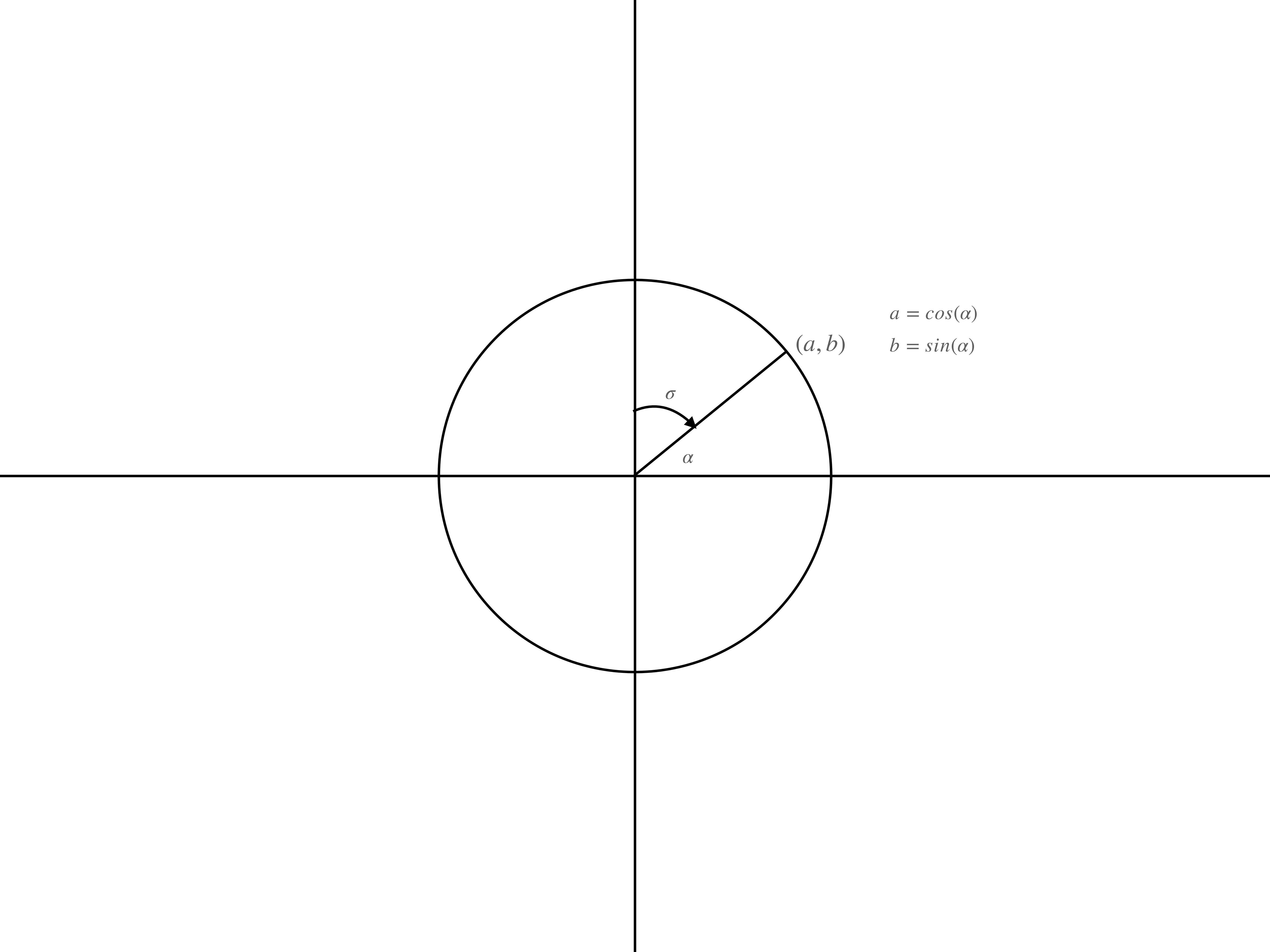


(a, b)
α
$a = cos(\alpha)$
$b = sin(\alpha)$



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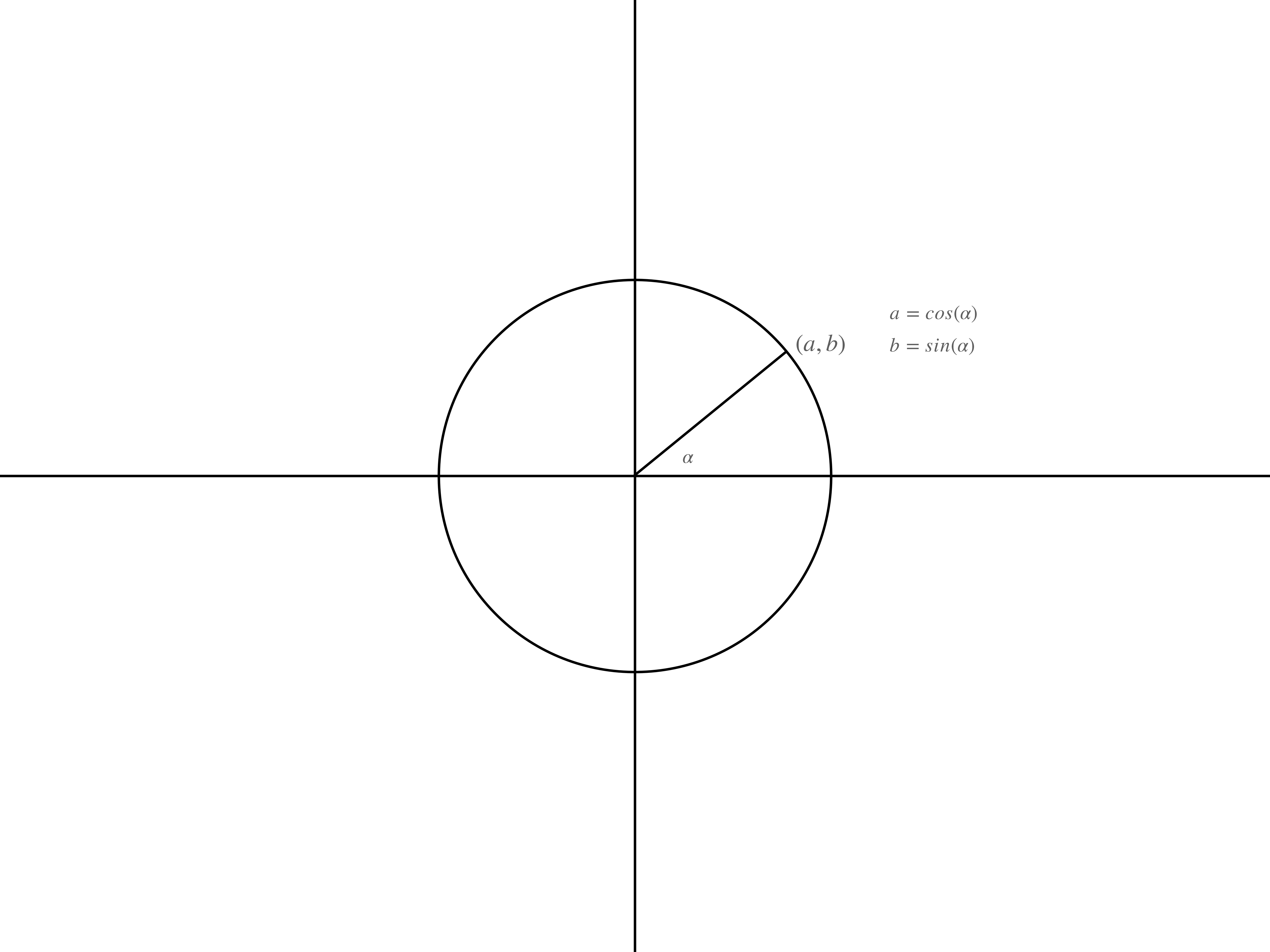
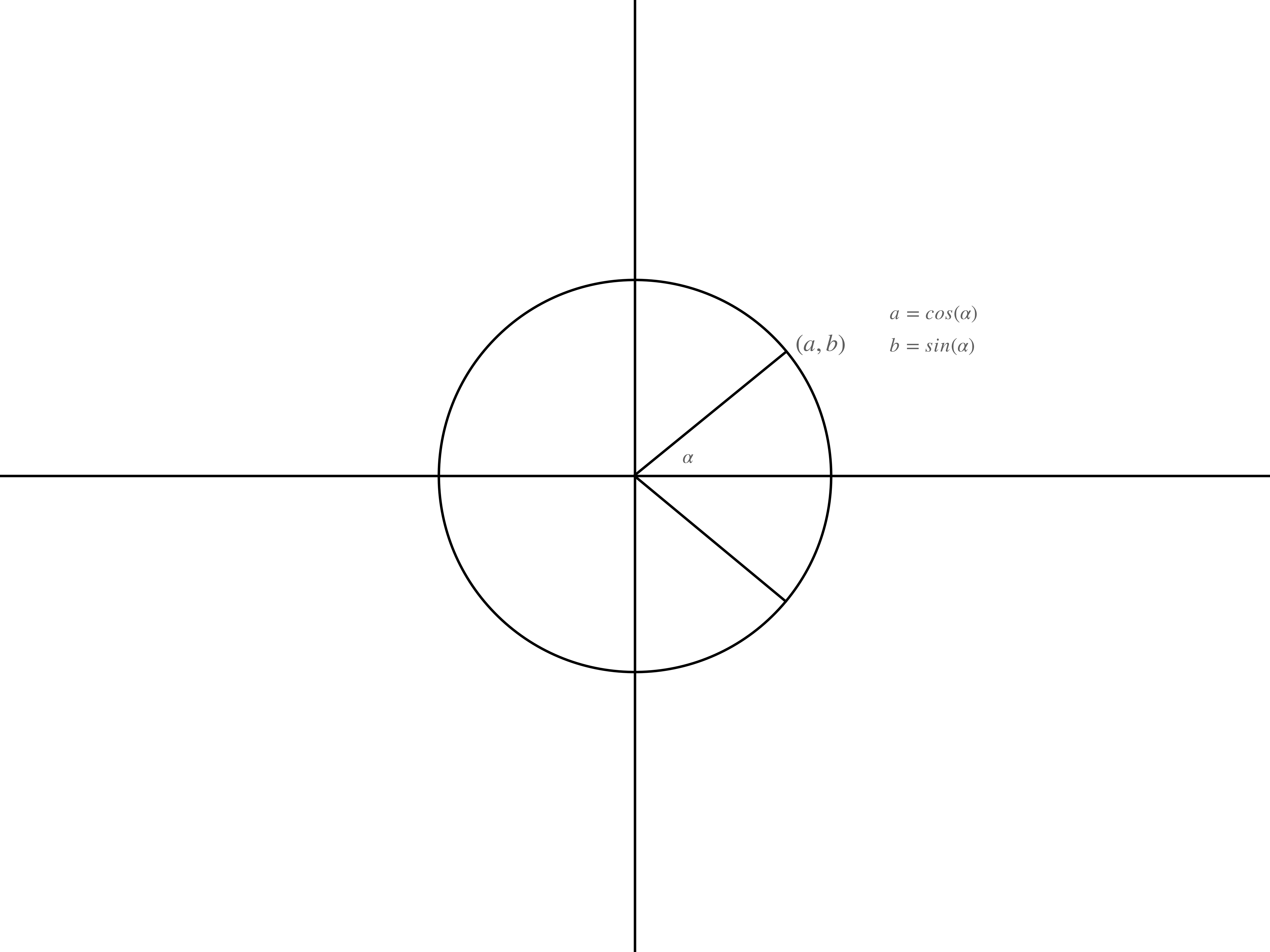
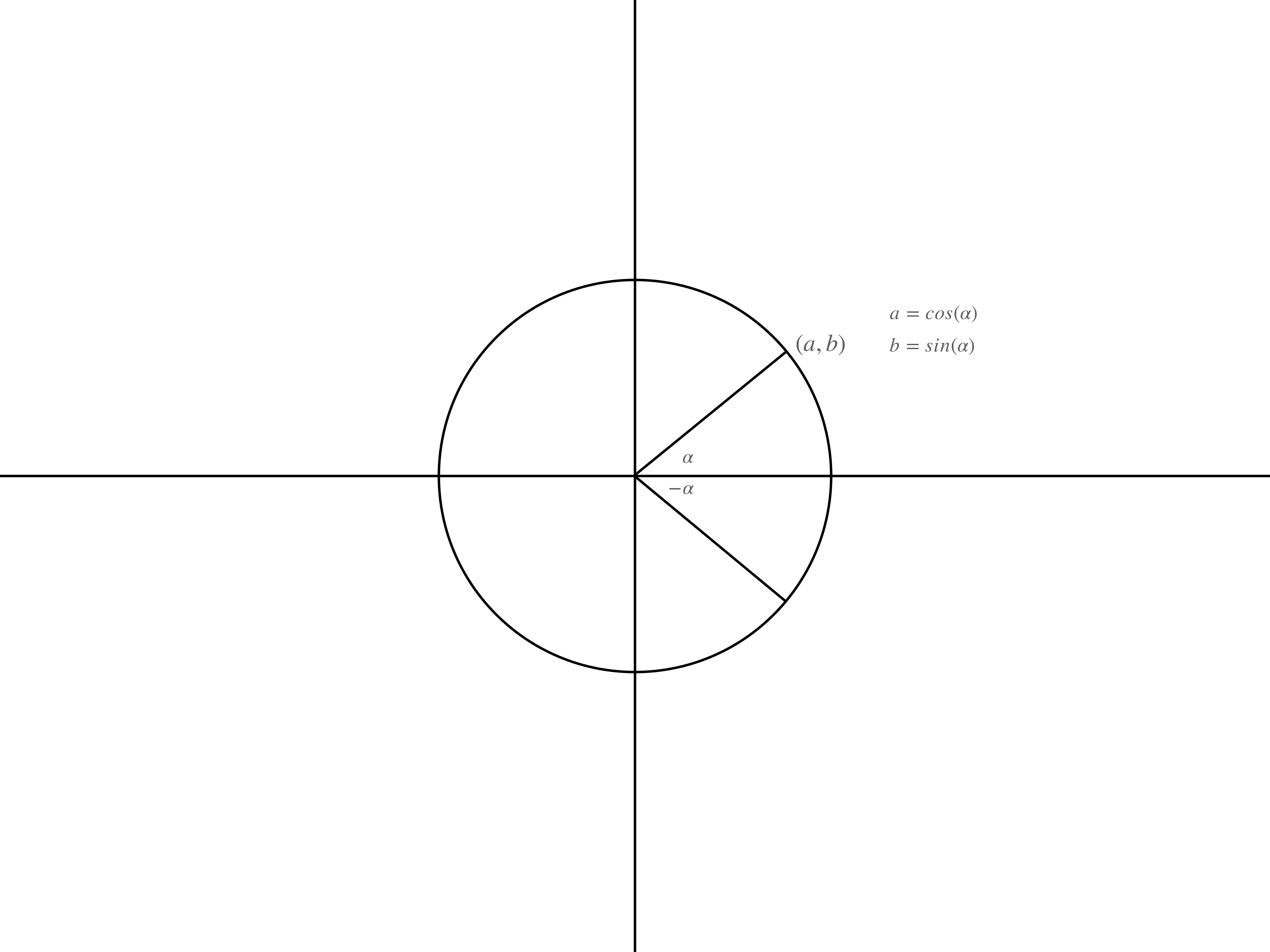
$a = cos(\alpha)$
$b = sin(\alpha)$
σ
α

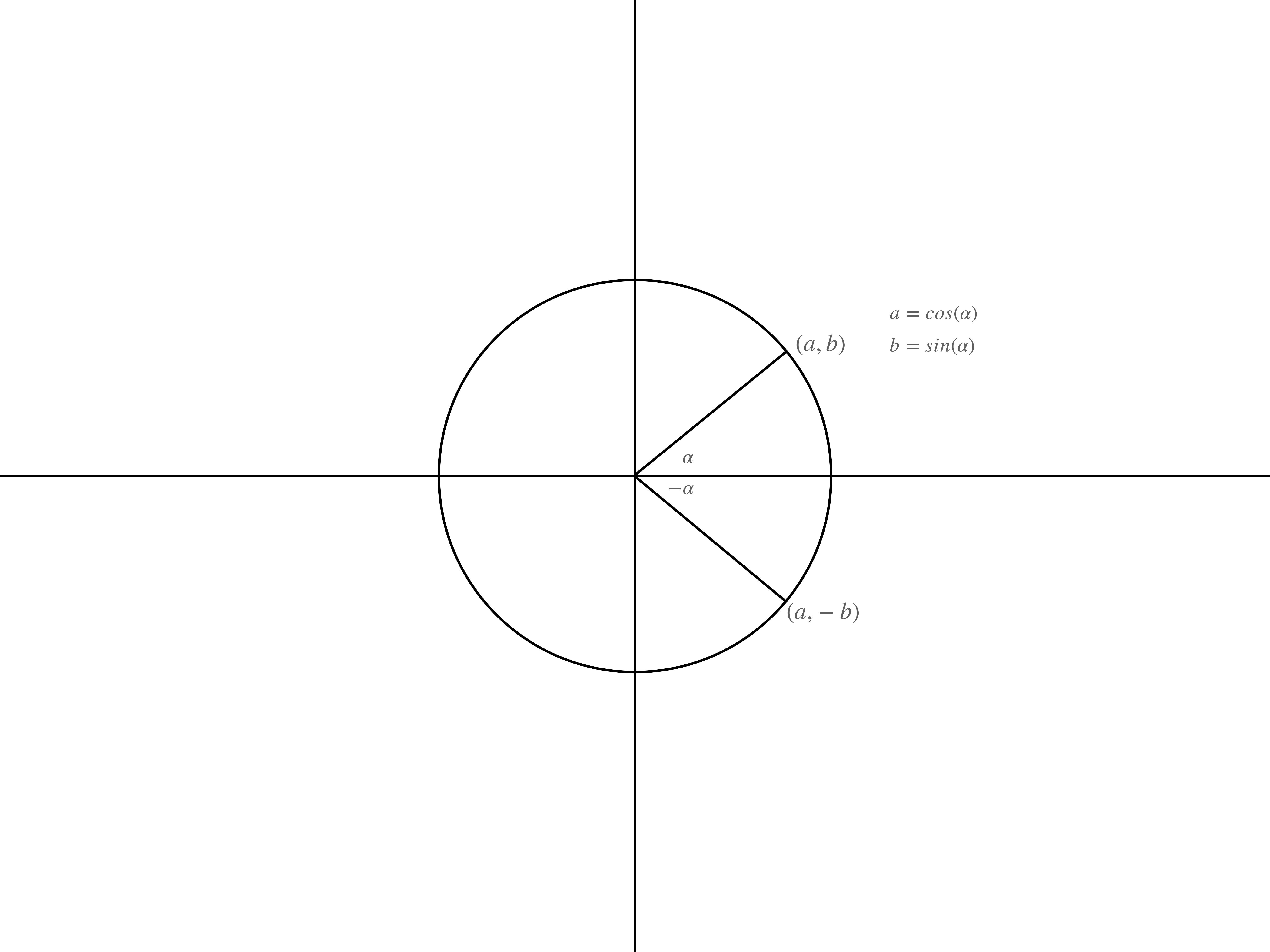


$a = cos(\alpha)$
$b = sin(\alpha)$
(a, b)
α

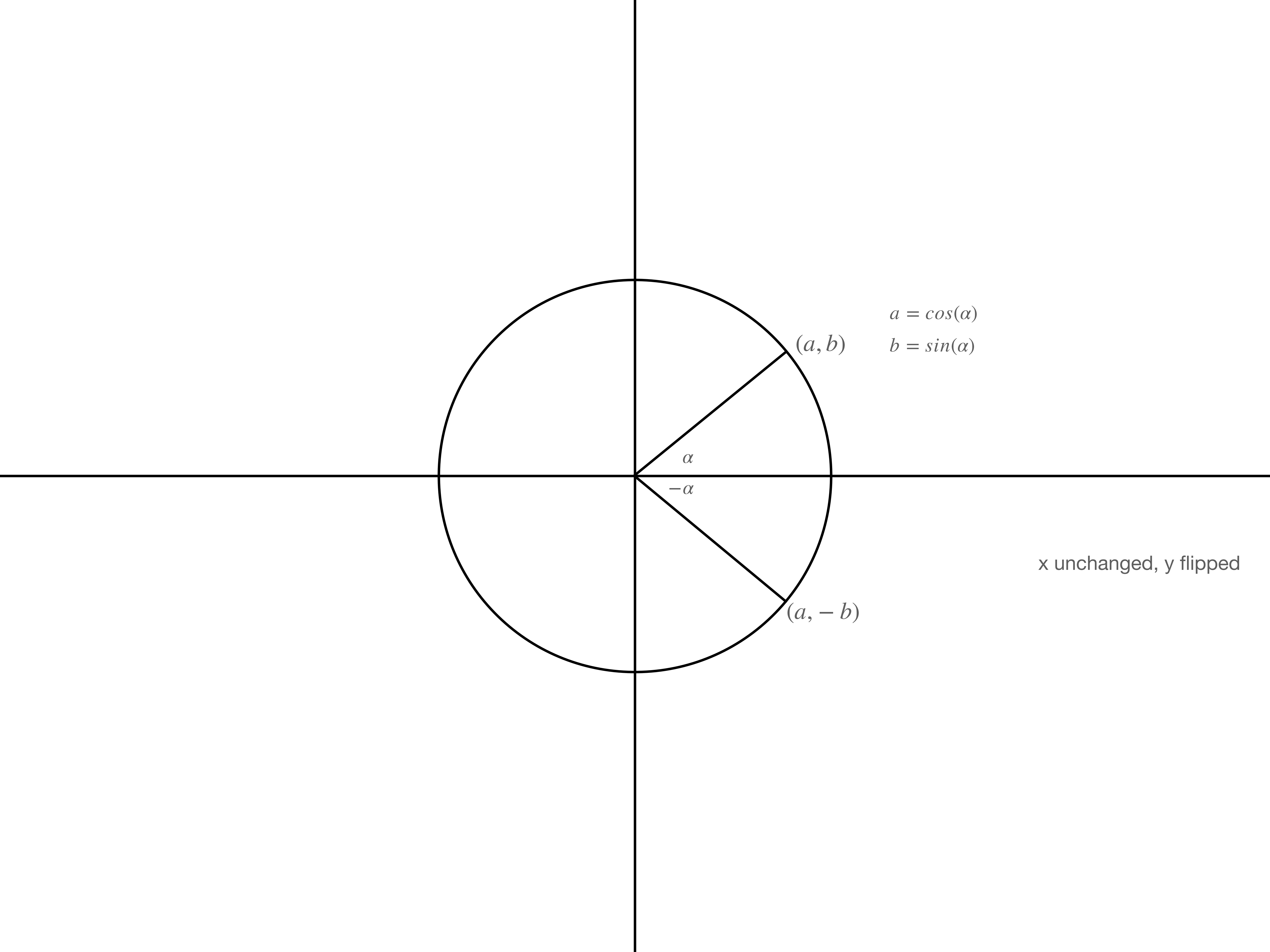

(a, b)
α
a = cos(α)
b = sin(α)



(a, b)
a = cos(α)
b = sin(α)
α
−α

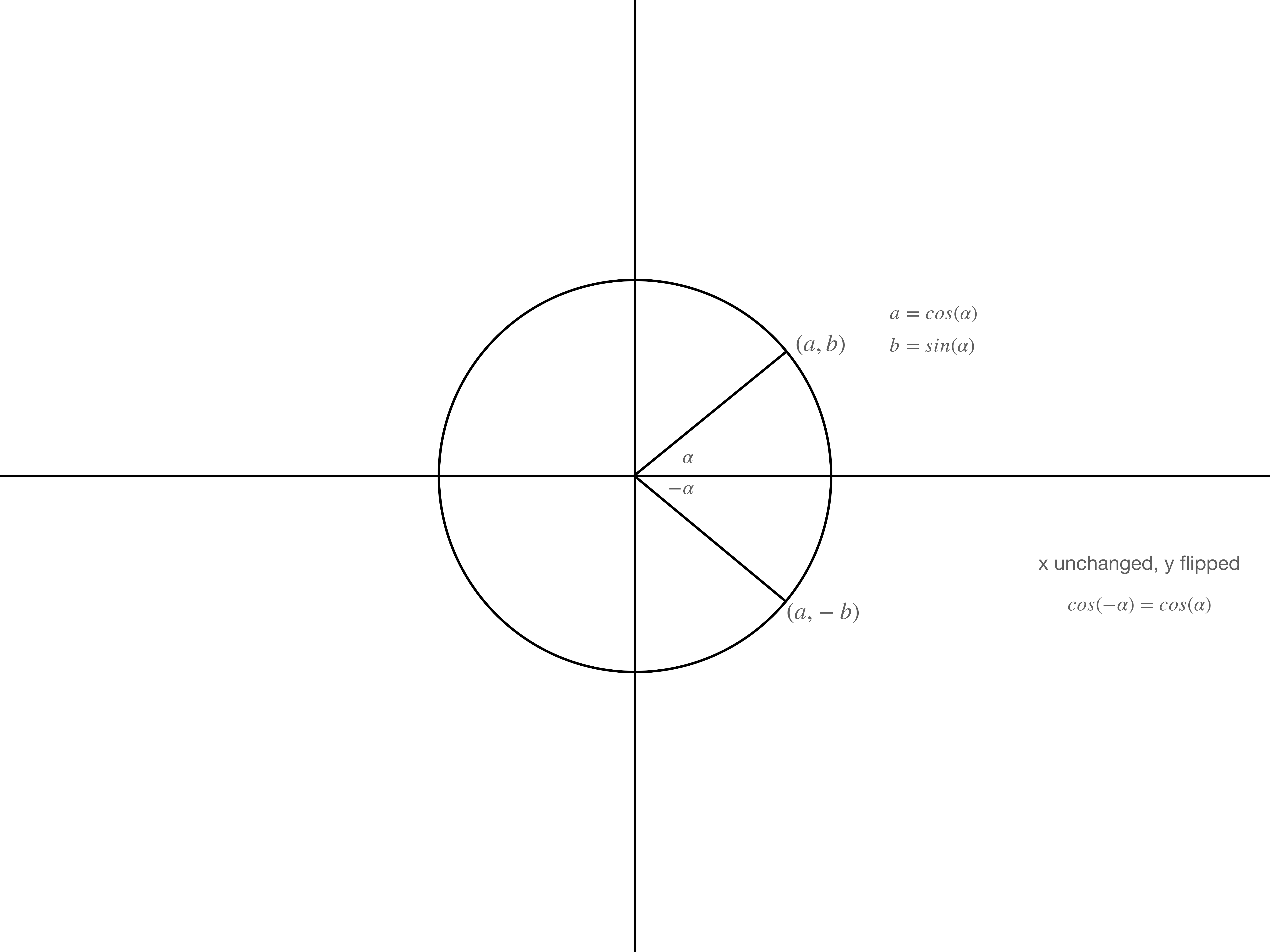


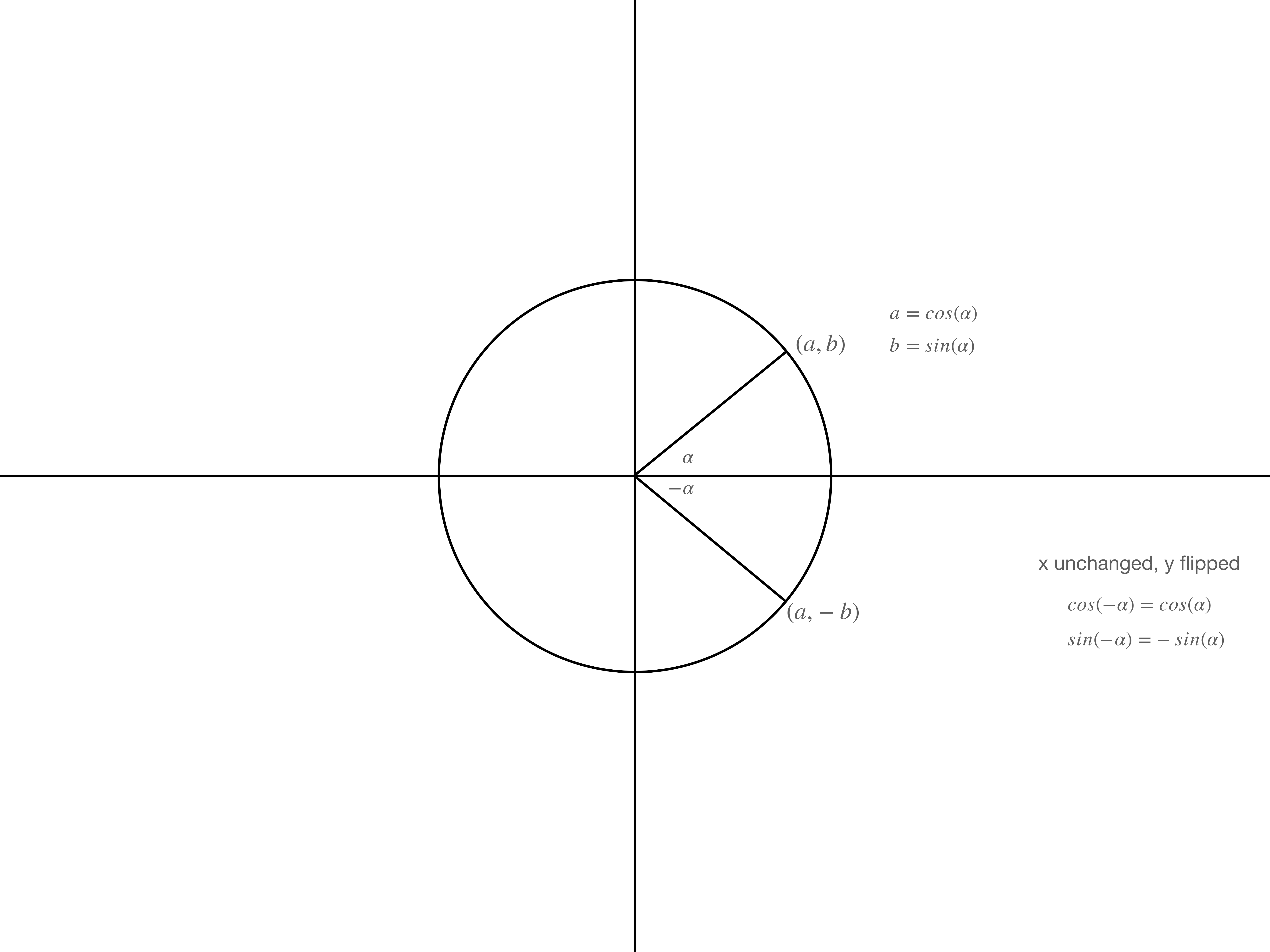
(a, b)
$a = cos(\alpha)$
$b = sin(\alpha)$
α
$-\alpha$
$(a, -b)$



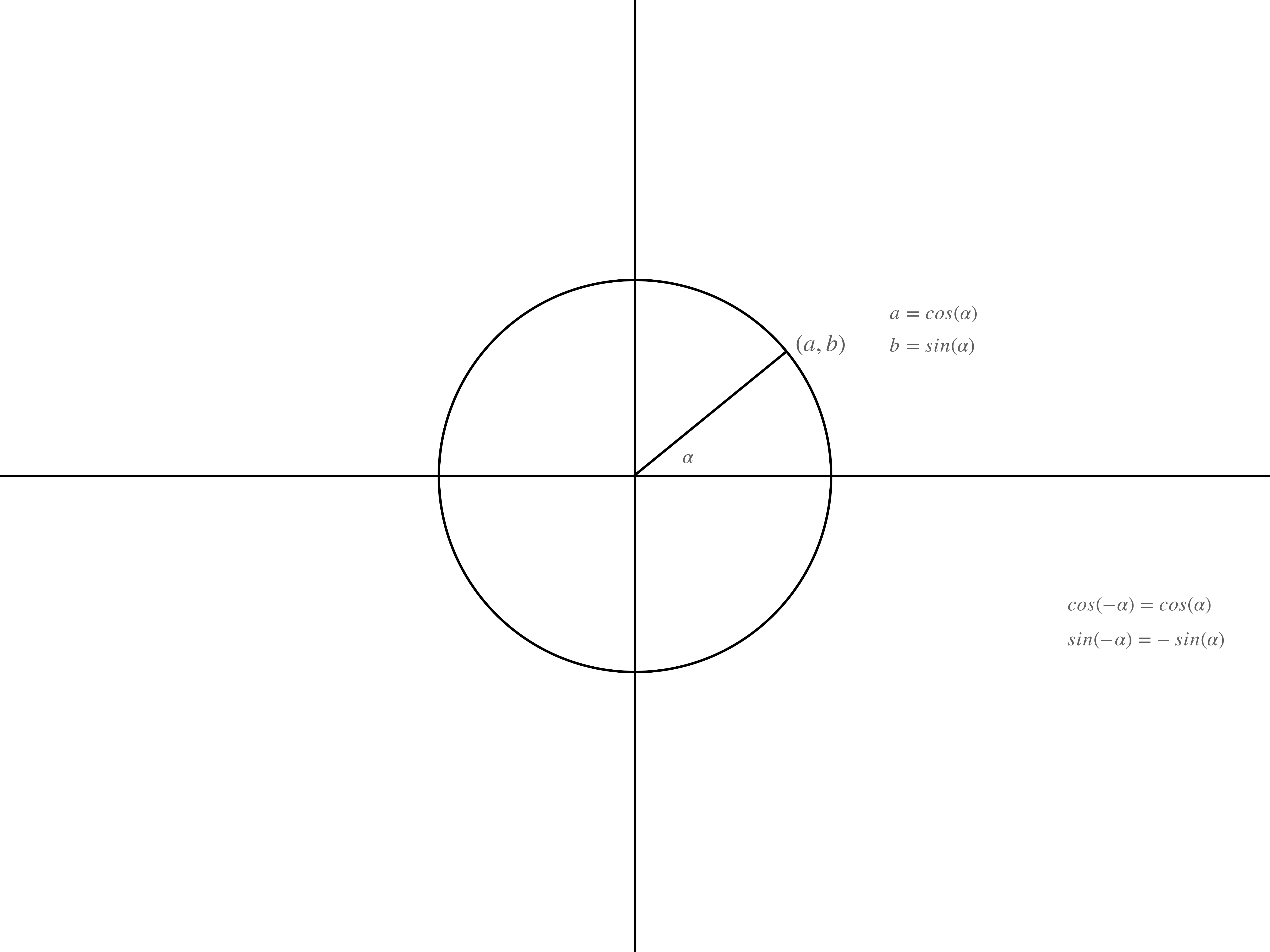
$a = cos(\alpha)$
$b = sin(\alpha)$

x unchanged, y flipped

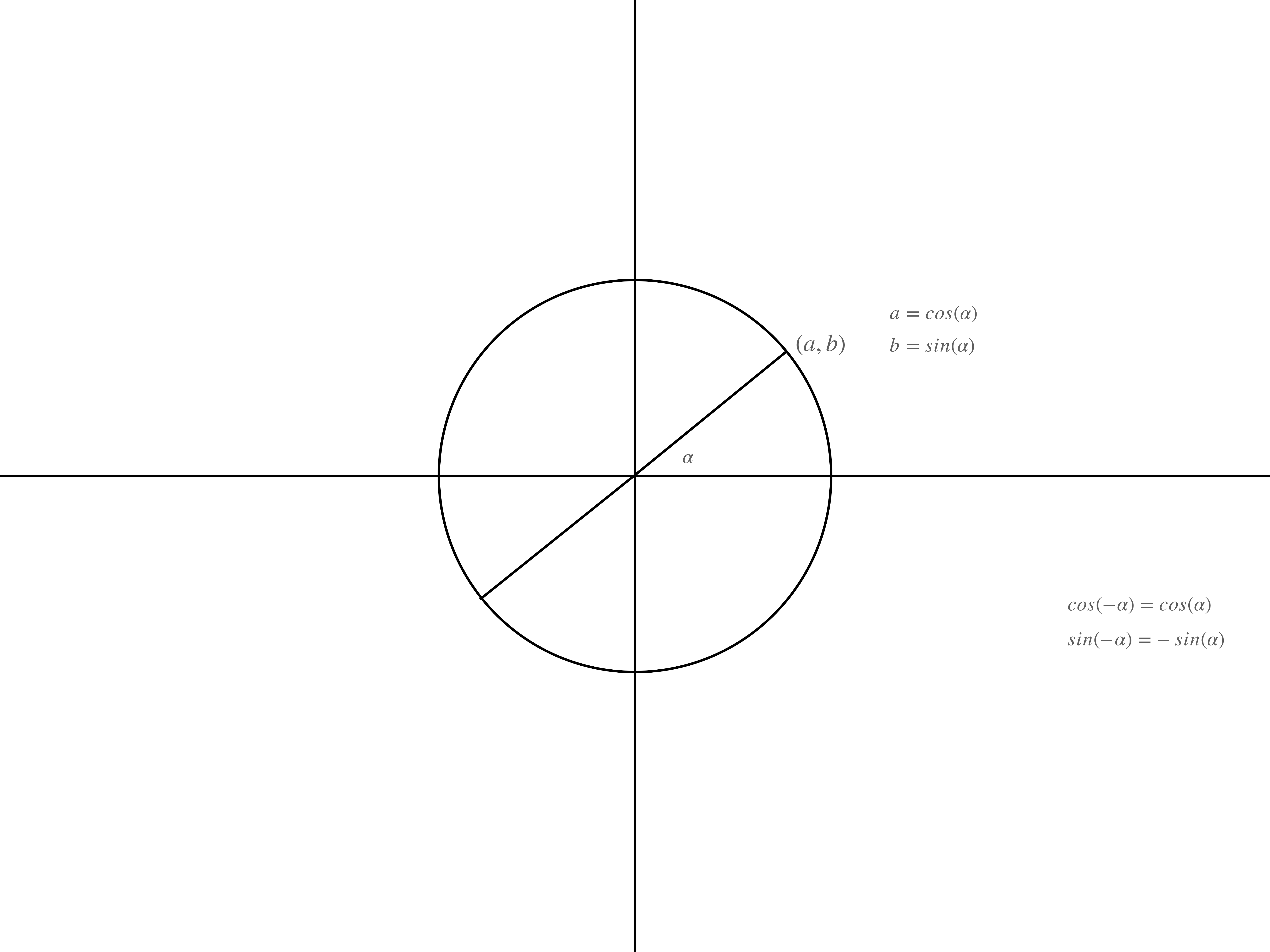

$a = cos(\alpha)$
$b = sin(\alpha)$
(a, b)
α
$-\alpha$
$(a, -b)$
x unchanged, y flipped
$cos(-\alpha) = cos(\alpha)$



(a, b)
$a = cos(\alpha)$
$b = sin(\alpha)$
α
$-\alpha$
$(a, -b)$
x unchanged, y flipped
$cos(-\alpha) = cos(\alpha)$
$sin(-\alpha) = -sin(\alpha)$



(a, b)
α
a = cos(α)
b = sin(α)
cos(−α) = cos(α)
sin(−α) = − sin(α)



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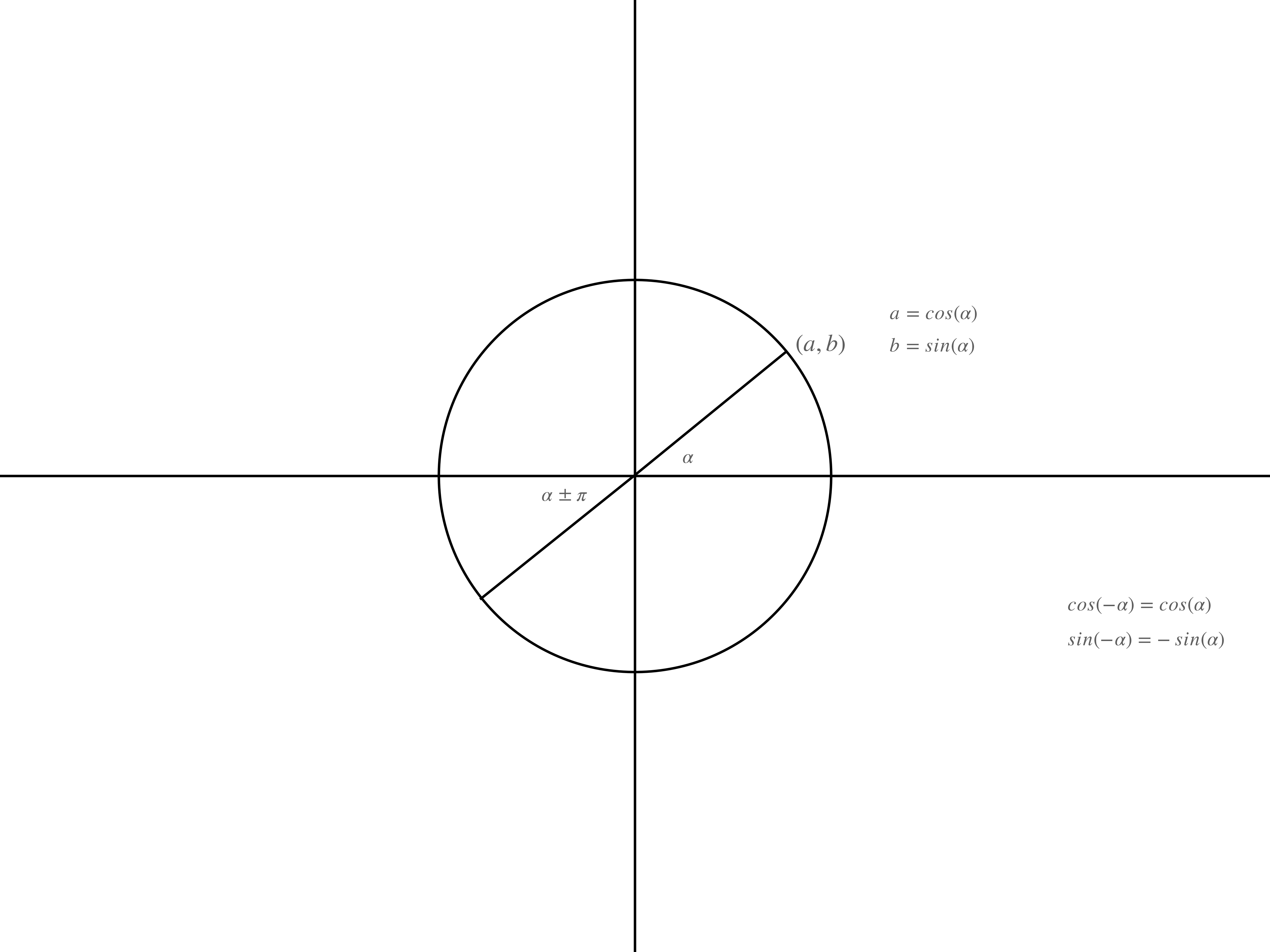
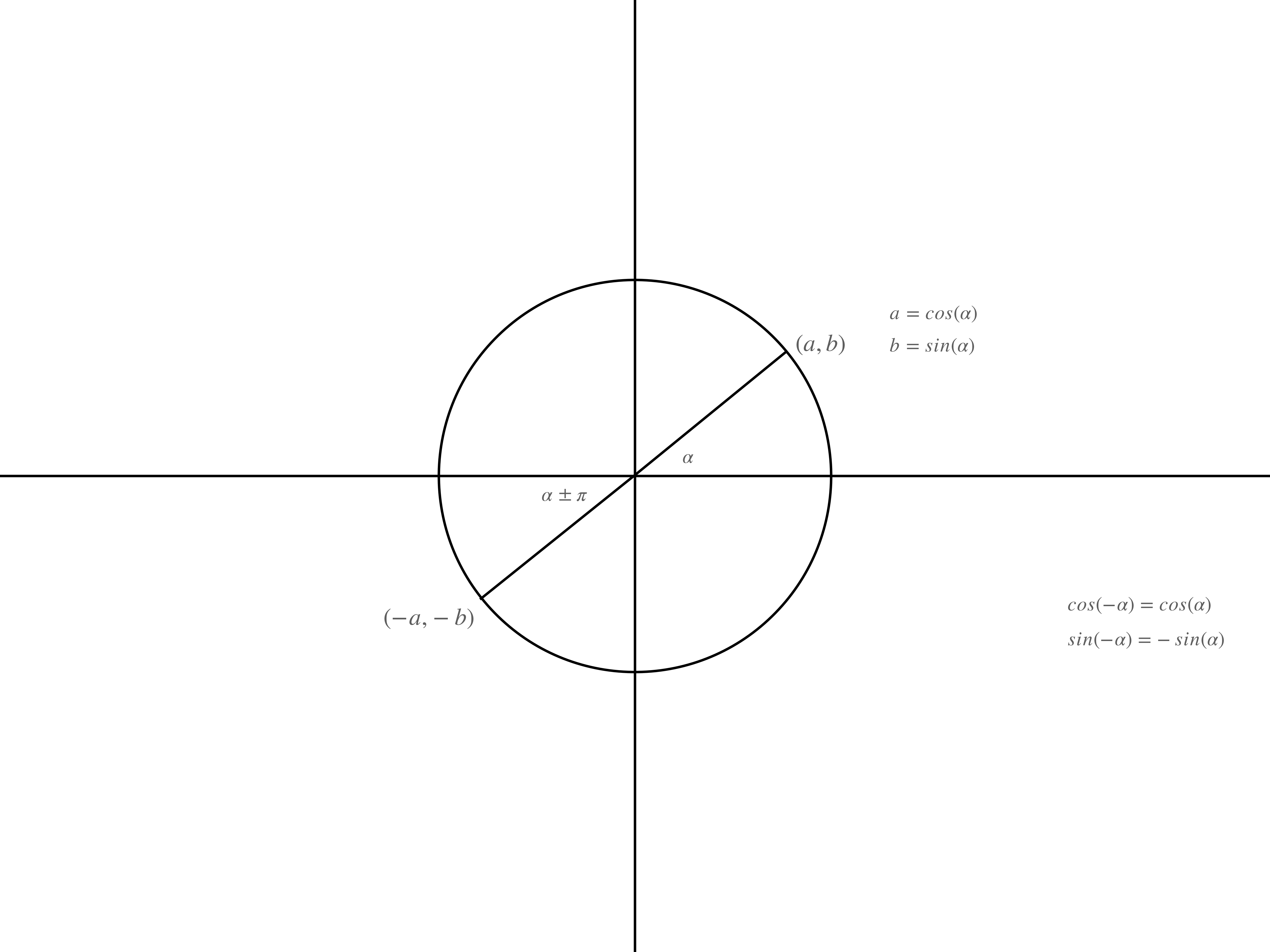

(a, b)
α
$\alpha \pm \pi$
$a = cos(\alpha)$
$b = sin(\alpha)$
$cos(-\alpha) = cos(\alpha)$
$sin(-\alpha) = - sin(\alpha)$



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α
$\alpha \pm \pi$
$(-a, -b)$
$cos(-\alpha) = cos(\alpha)$
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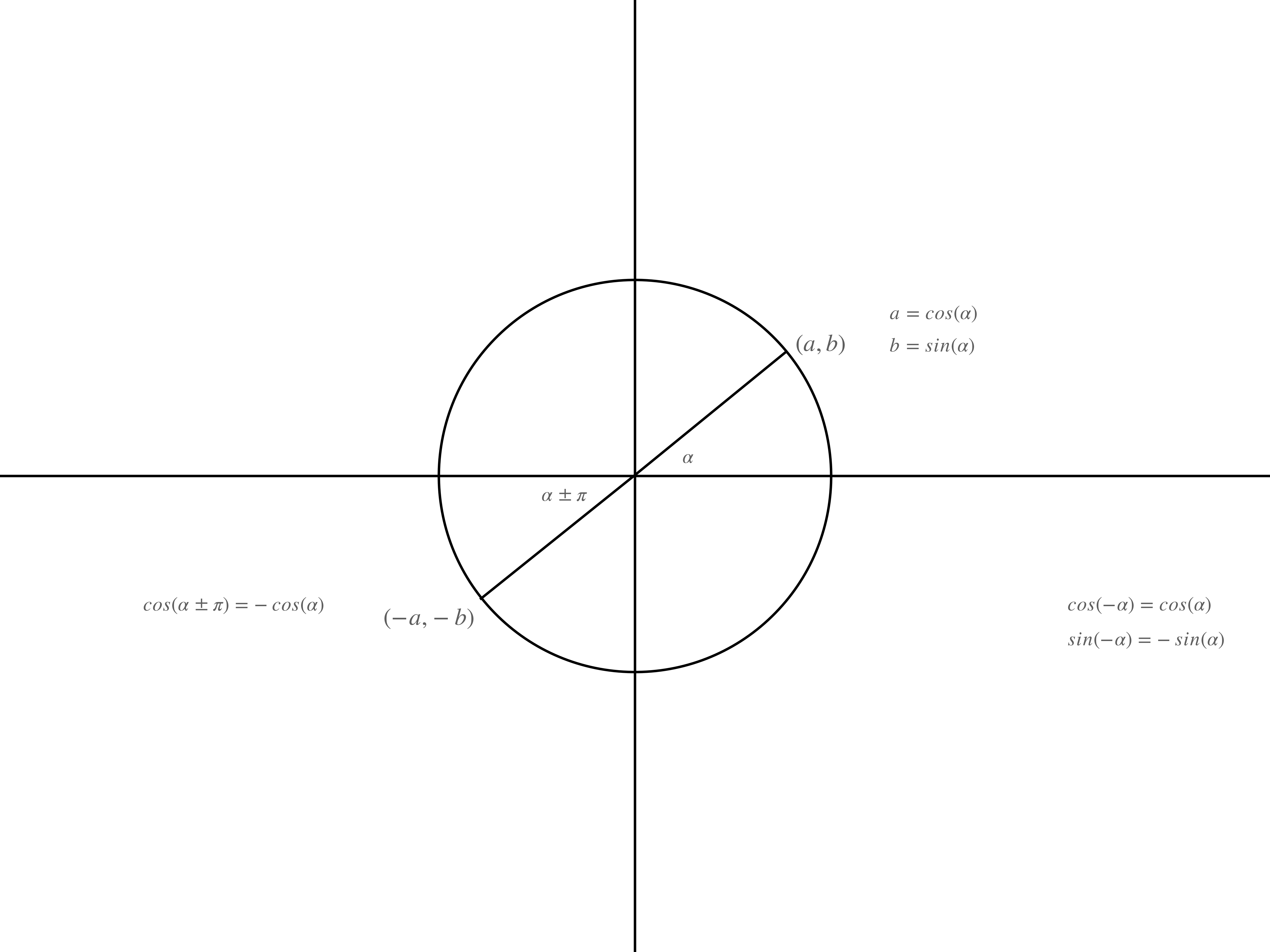
$a = cos(\alpha)$
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(a, b)
α
$\alpha \pm \pi$
$cos(\alpha \pm \pi) = -\ cos(\alpha)$
$(-a, -\ b)$
$cos(-\alpha) = cos(\alpha)$
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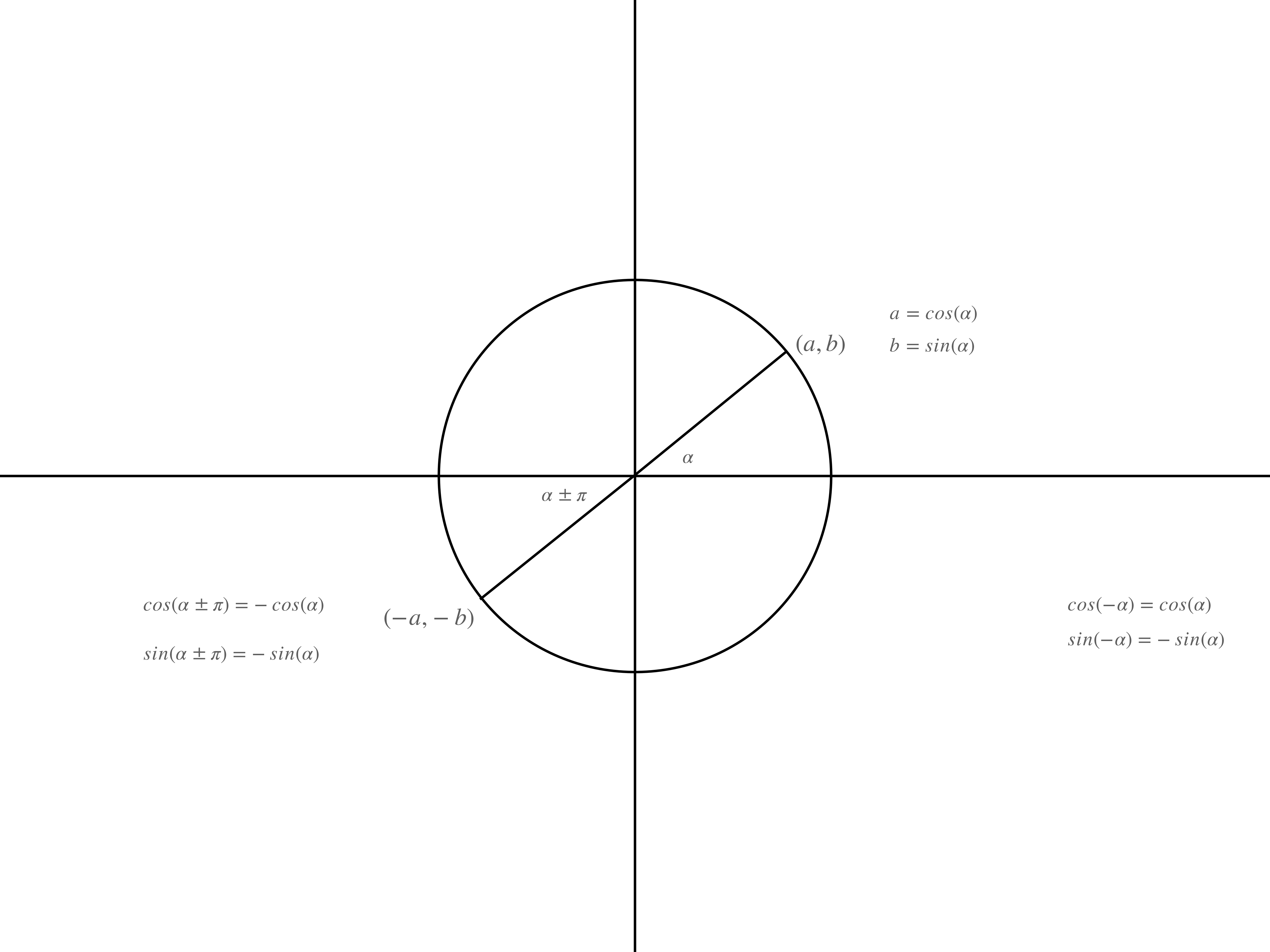

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(a, b)
α
$\alpha \pm \pi$
$(-a, -b)$
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$cos(-\alpha) = cos(\alpha)$
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$$cos(\beta - \alpha) = cos(\beta)cos(\alpha)) + sin(\beta)sin(\alpha)$$

Trick 1: use -a to get additive angles

$$cos(\beta - \alpha) = cos(\beta)cos(\alpha)) + sin(\beta)sin(\alpha)$$

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$$cos(\beta + \alpha) = cos(\beta)cos(\alpha) - sin(\beta)sin(\alpha)$$

Trick 2: find cos of complementary angle of a: (90-a)

$$cos(\beta - \alpha) = cos(\beta)cos(\alpha)) + sin(\beta)sin(\alpha)$$

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Trick 2: find cos of complementary angle of a: (90-a)

$$cos(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2})cos(\alpha) + sin(\frac{\pi}{2})sin(\alpha)$$

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$$cos(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2})cos(\alpha) + sin(\frac{\pi}{2})sin(\alpha)$$

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Trick 3: reverse and find sin of complementary angle (90-a)

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$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

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$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

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$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

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Now find sin of combined angles

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$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

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Now find sin of combined angles

$$sin(\beta + \alpha) = cos\left(\frac{\pi}{2} - (\beta + \alpha)\right)$$

Trick 1: use -a to get additive angles

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$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

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Now find sin of combined angles

$$sin(\beta + \alpha) = cos\Big(\frac{\pi}{2} - (\beta + \alpha)\Big)$$

$$sin(\beta + \alpha) = cos\Big((\frac{\pi}{2} - \beta) - \alpha)\Big)$$

Trick 1: use -a to get additive angles

$$cos(\beta - (-\alpha)) = cos(\beta)cos(-\alpha) + sin(\beta)sin(-\alpha)$$

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$$cos(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2})cos(\alpha) + sin(\frac{\pi}{2})sin(\alpha)$$

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$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

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Now find sin of combined angles

$$sin(\beta + \alpha) = cos\left(\frac{\pi}{2} - (\beta + \alpha)\right)$$

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** don’t get misled - complementary angles are a trig concept in triangles **

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$$sin(\beta + \alpha) = cos\left((\frac{\pi}{2} - \beta) - \alpha)\right)$$

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$$sin(\beta + \alpha) = sin(\beta)cos(\alpha) + cos(\beta)sin(\alpha)$$

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Trick 3: reverse and find sin of complementary angle (90-a)

$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - (\frac{\pi}{2} - \alpha))$$

$$sin(\frac{\pi}{2} - \alpha) = cos(\frac{\pi}{2} - \frac{\pi}{2} + \alpha)$$

$$sin(\frac{\pi}{2} - \alpha) = cos(\alpha)$$

** don’t get misled - complementary angles are a trig concept in triangles **

** the pi/2 lead/lag is true in the general trig concept of functions at all x **

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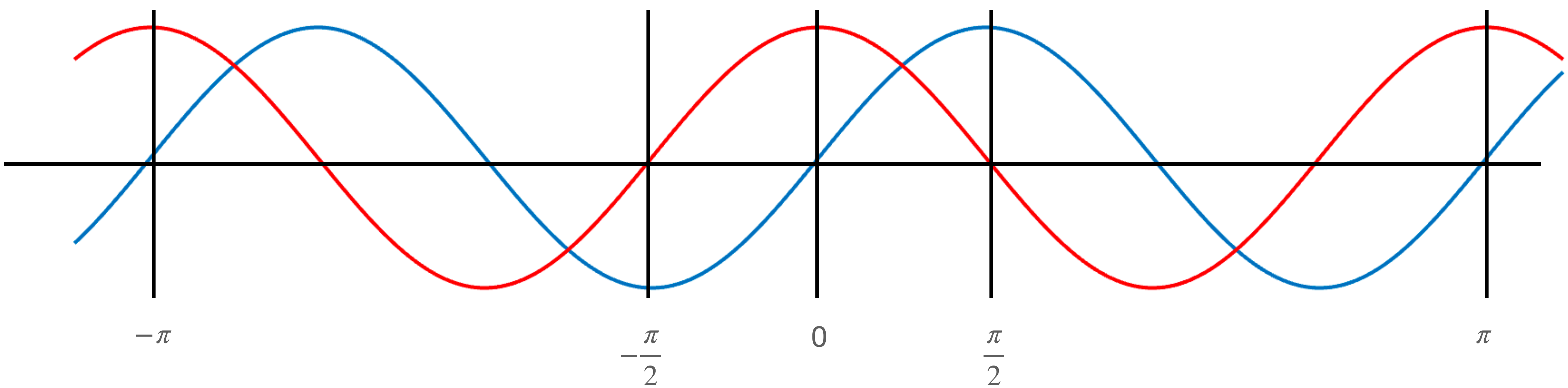

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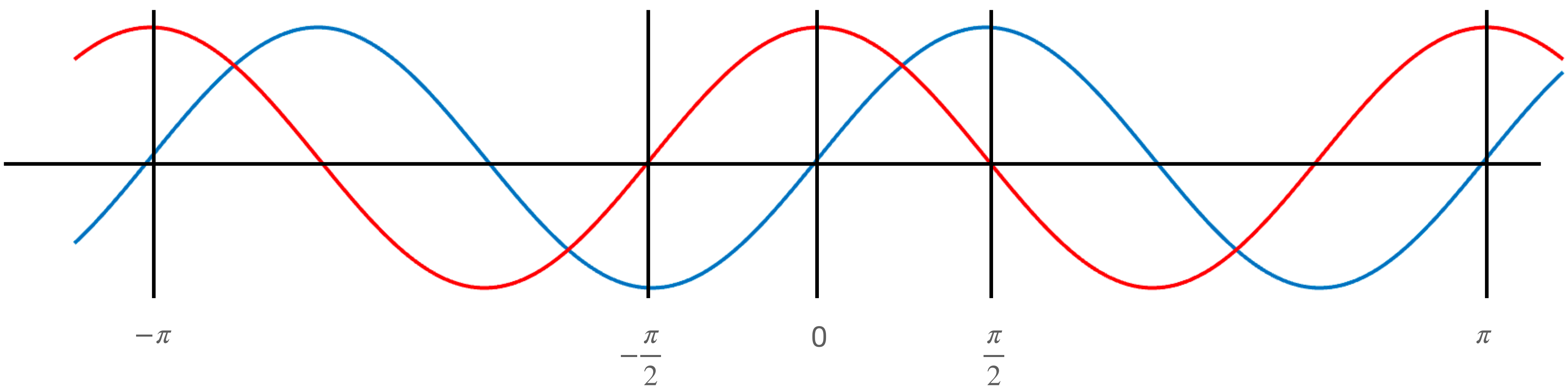

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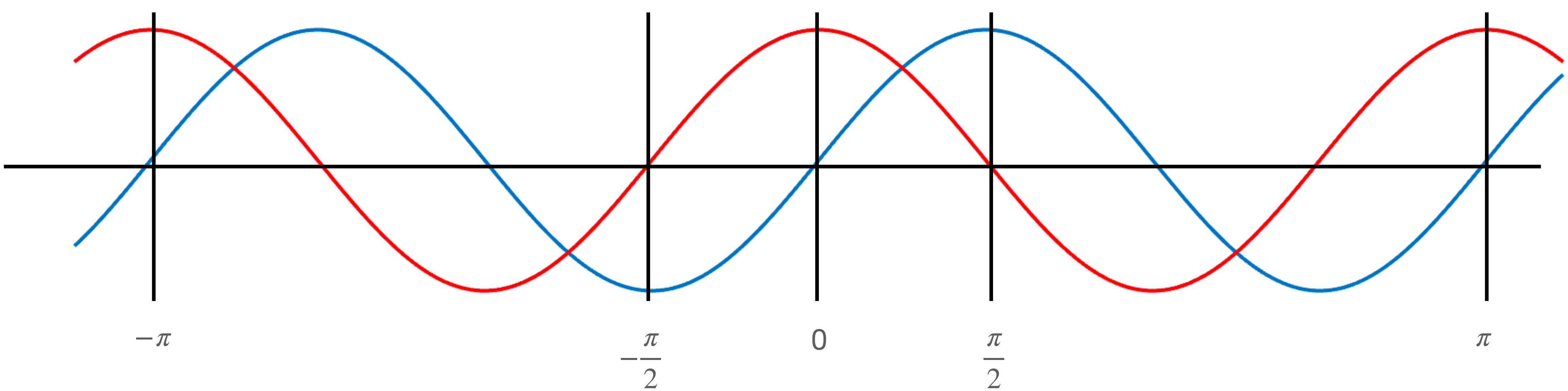

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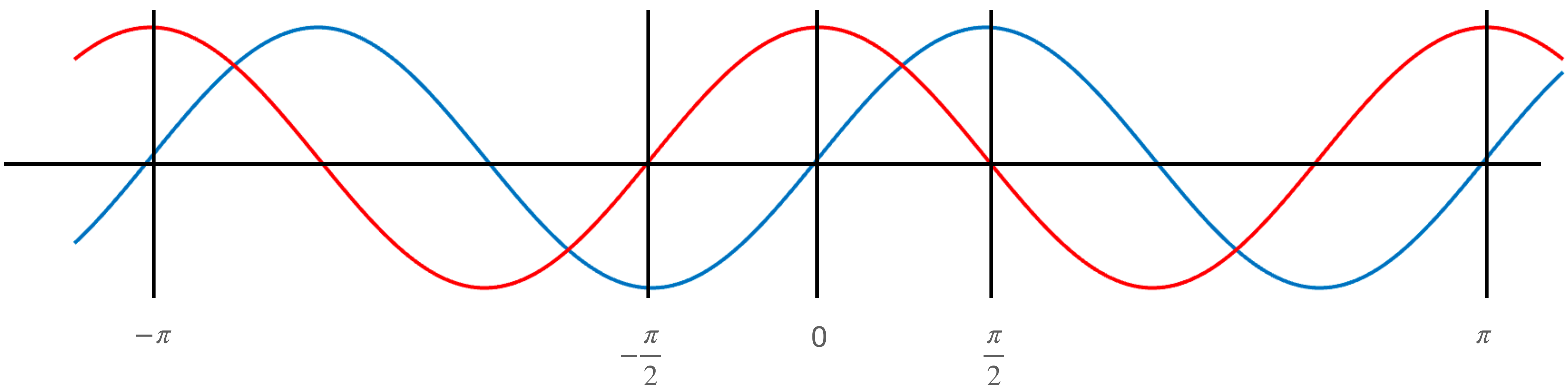

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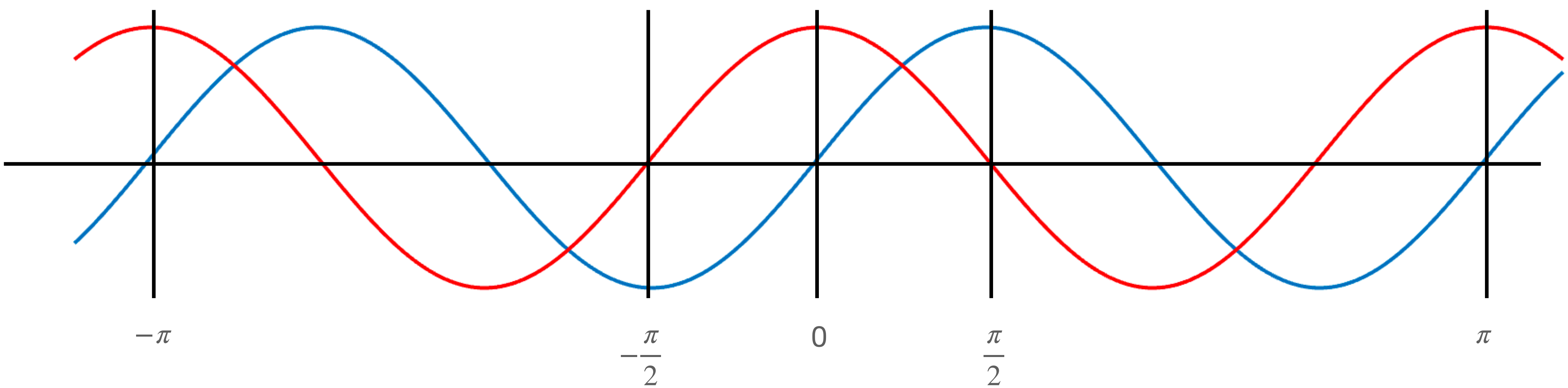

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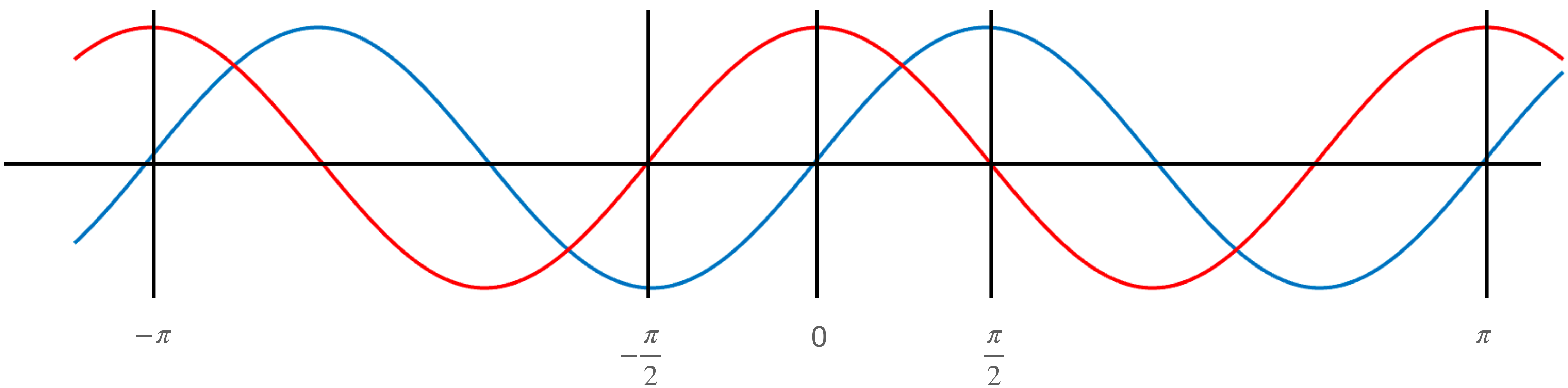

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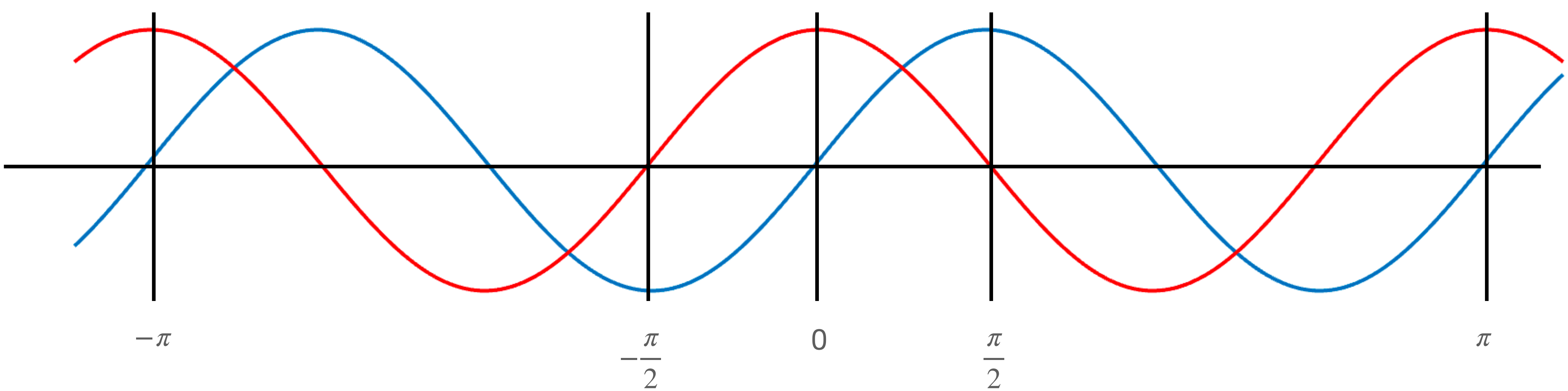

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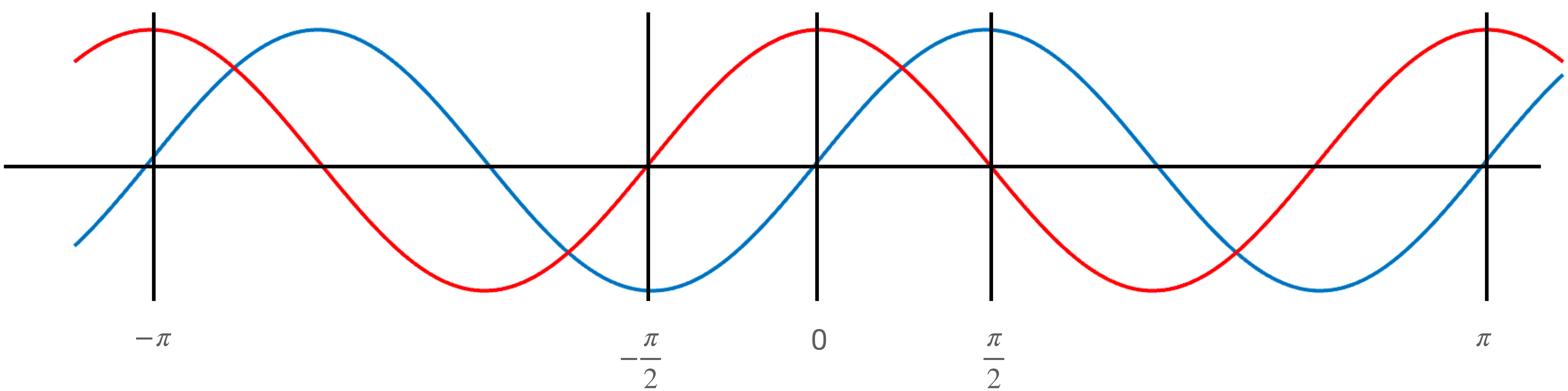

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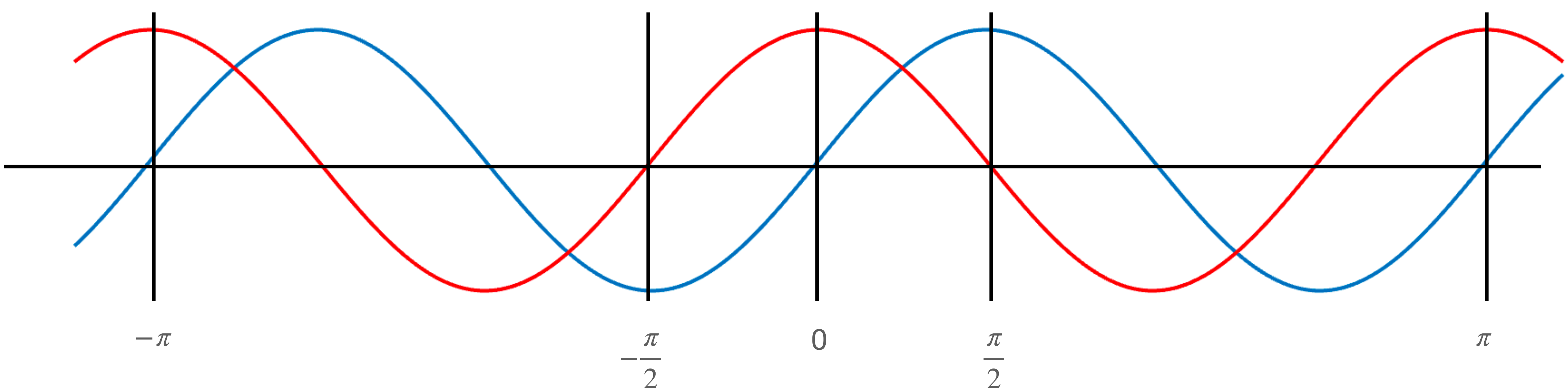

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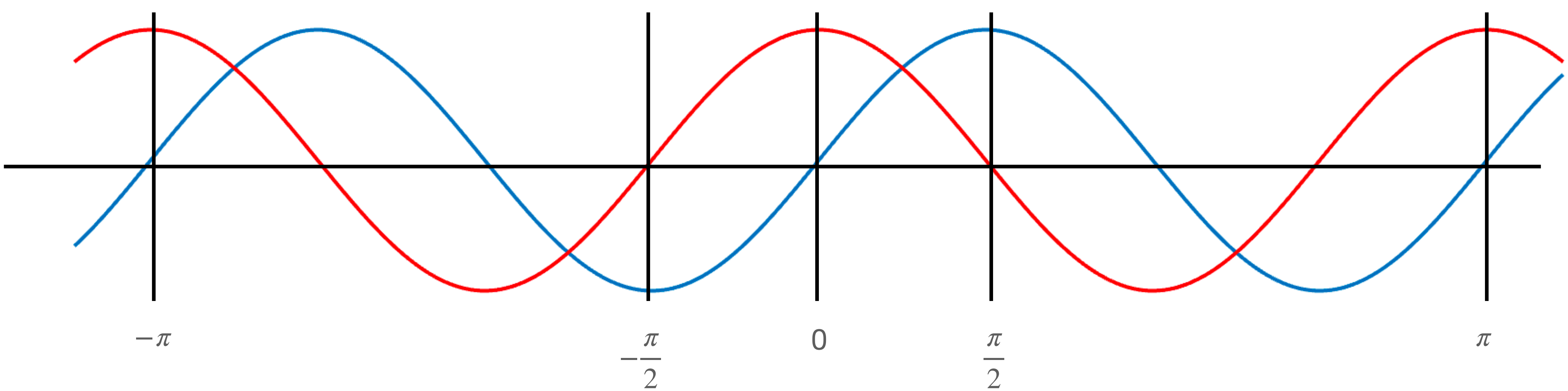

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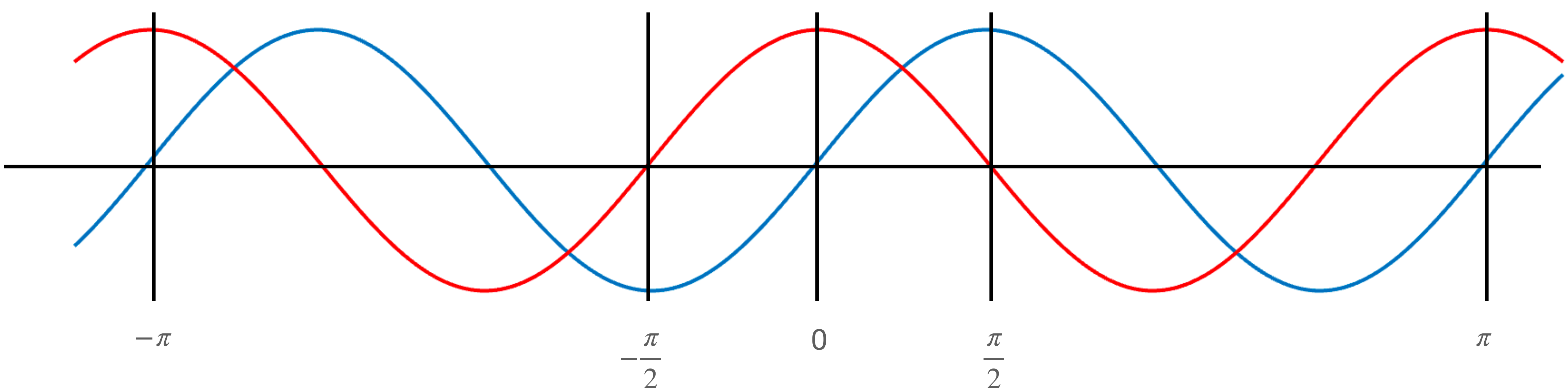

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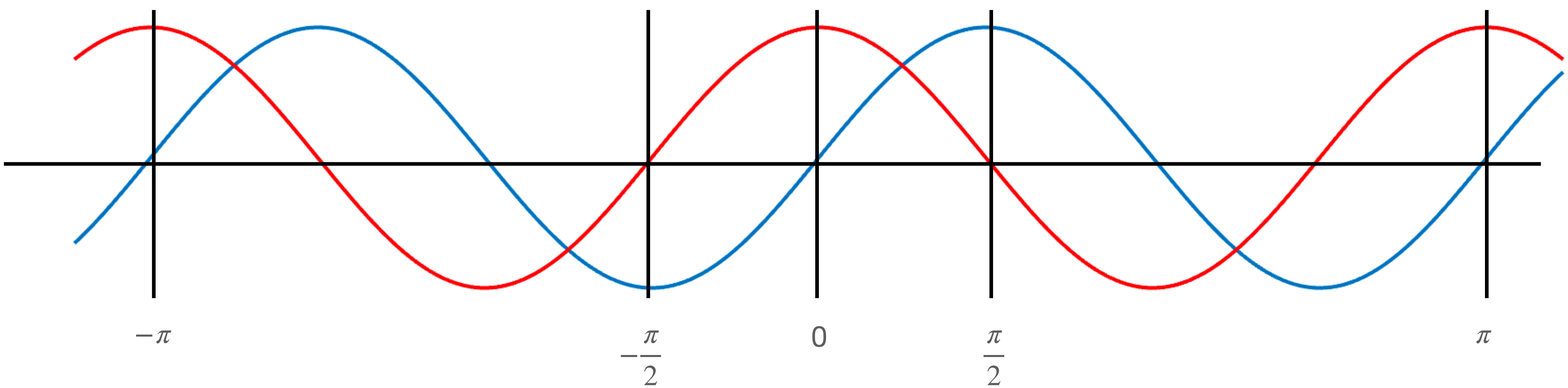

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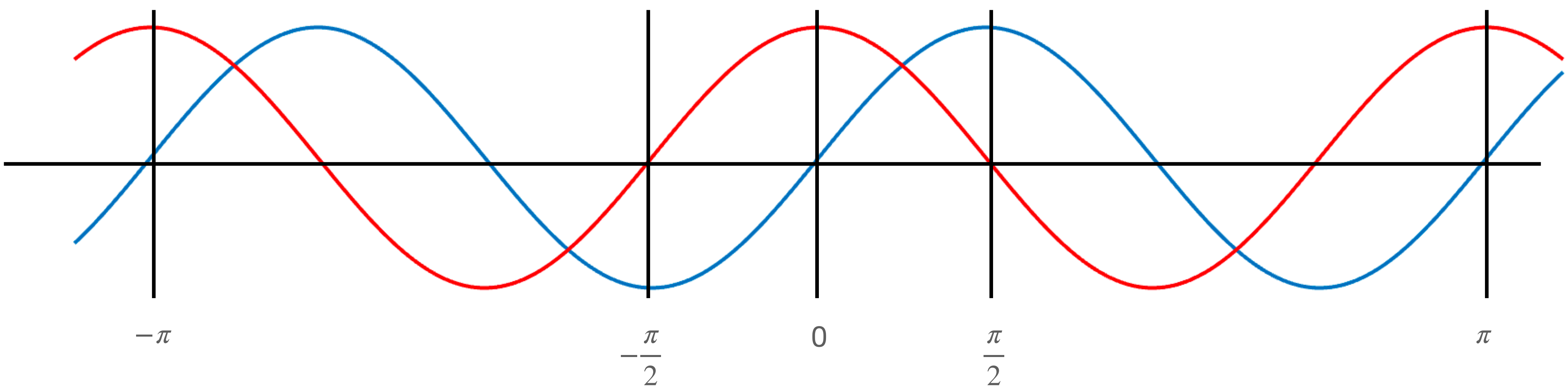

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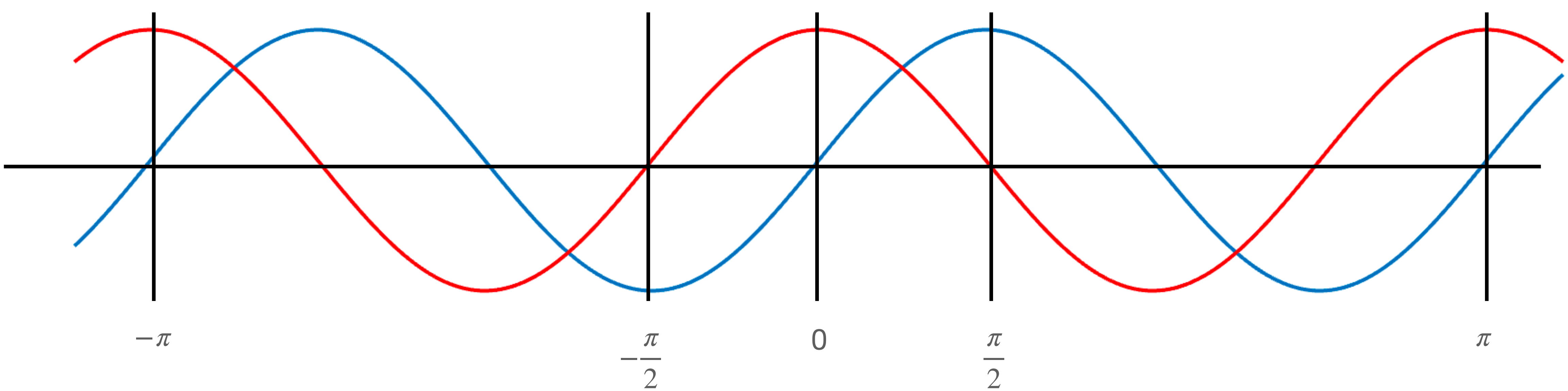

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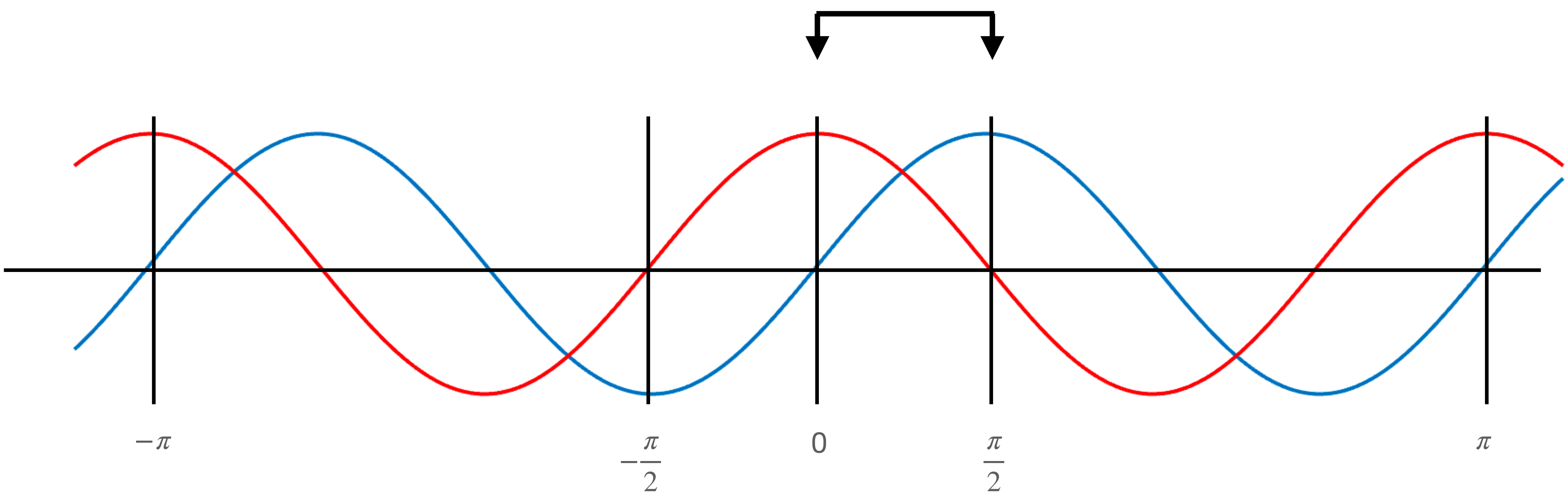

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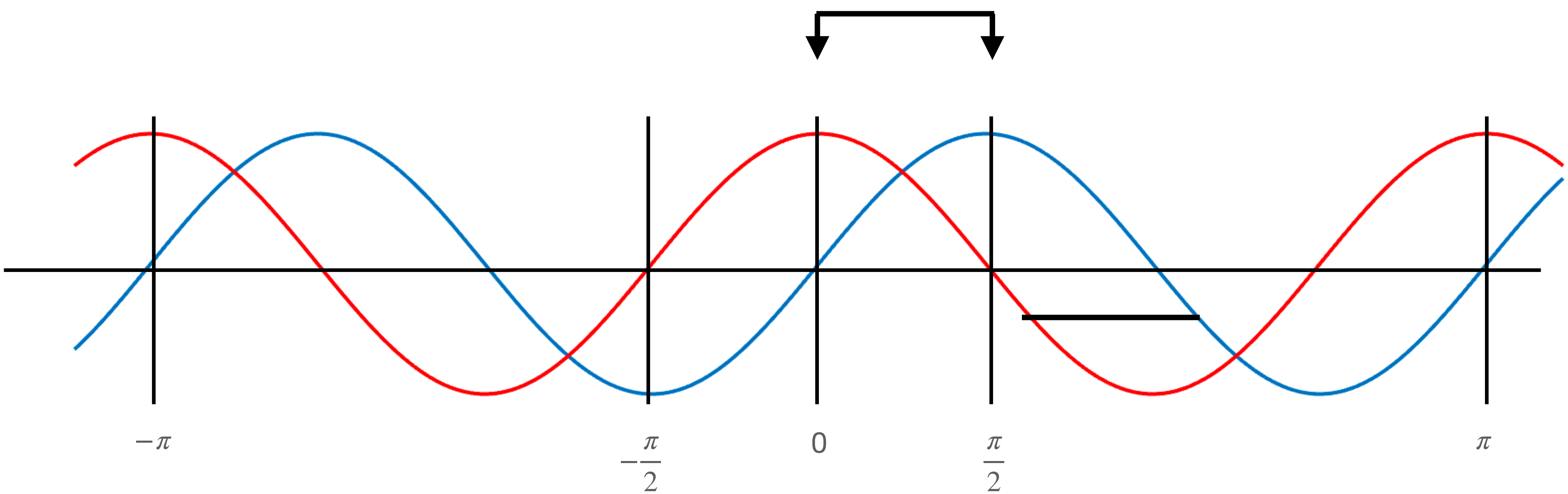

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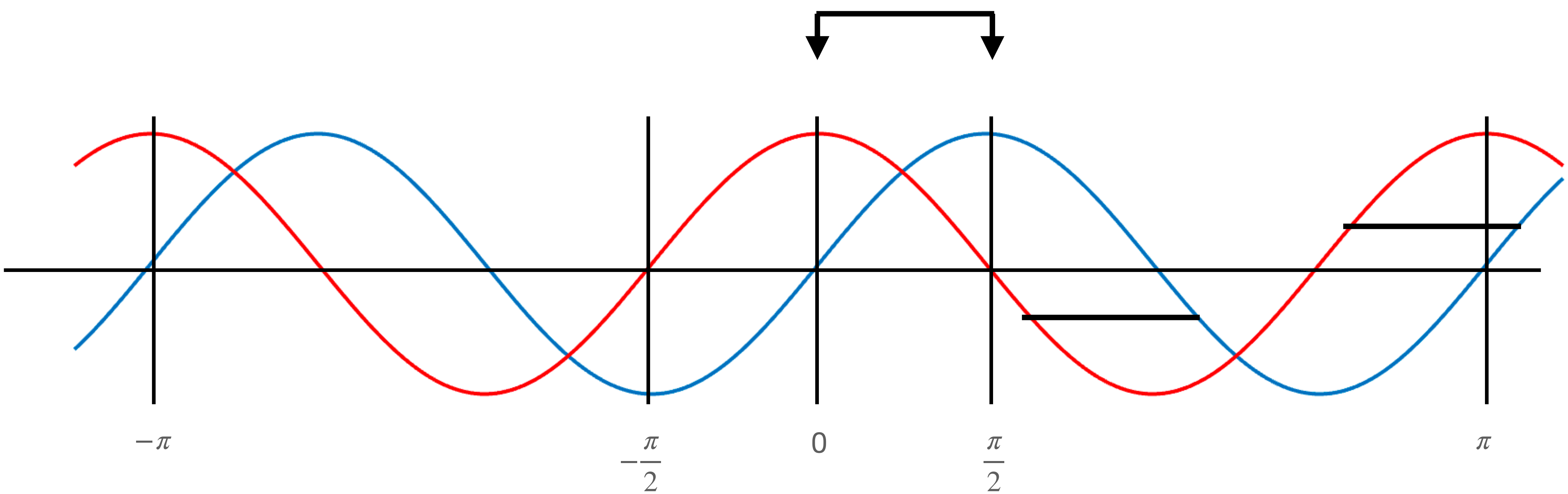

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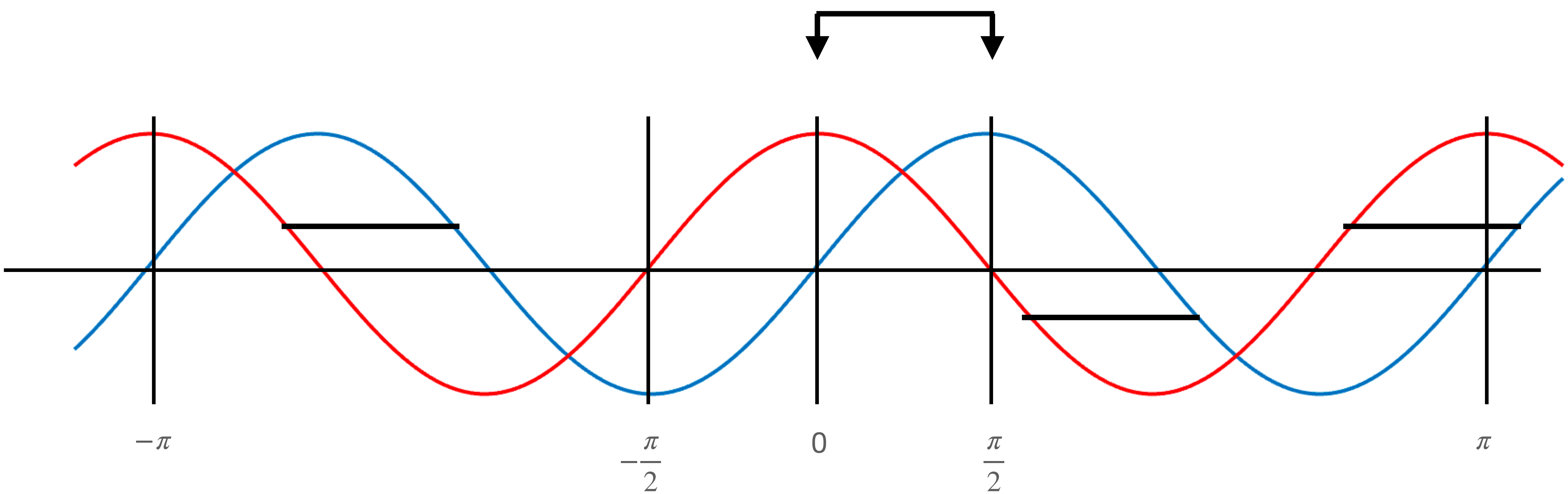

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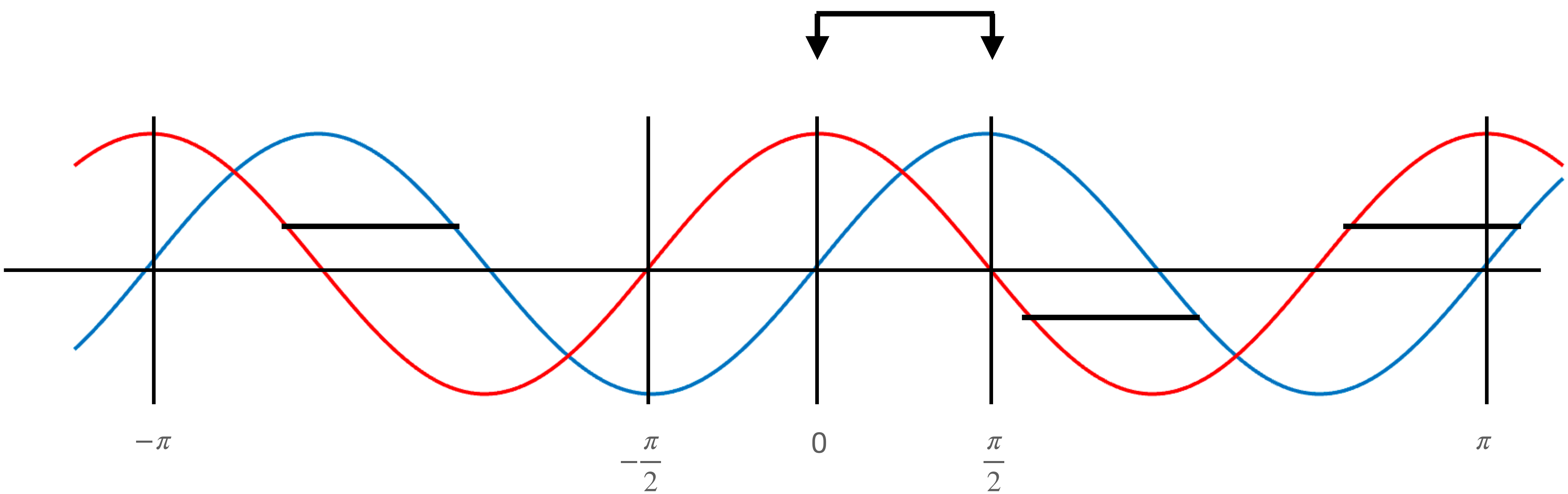

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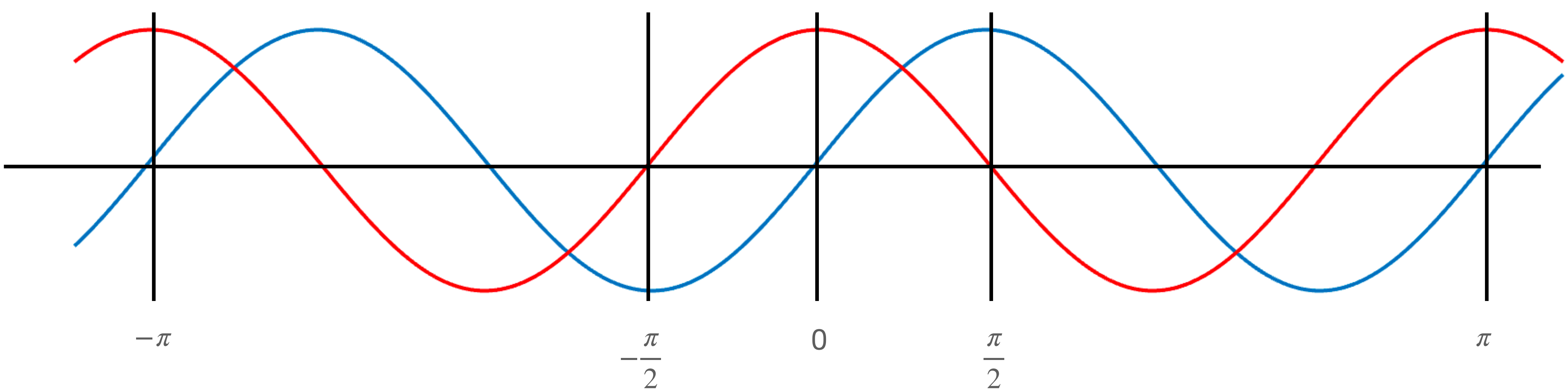

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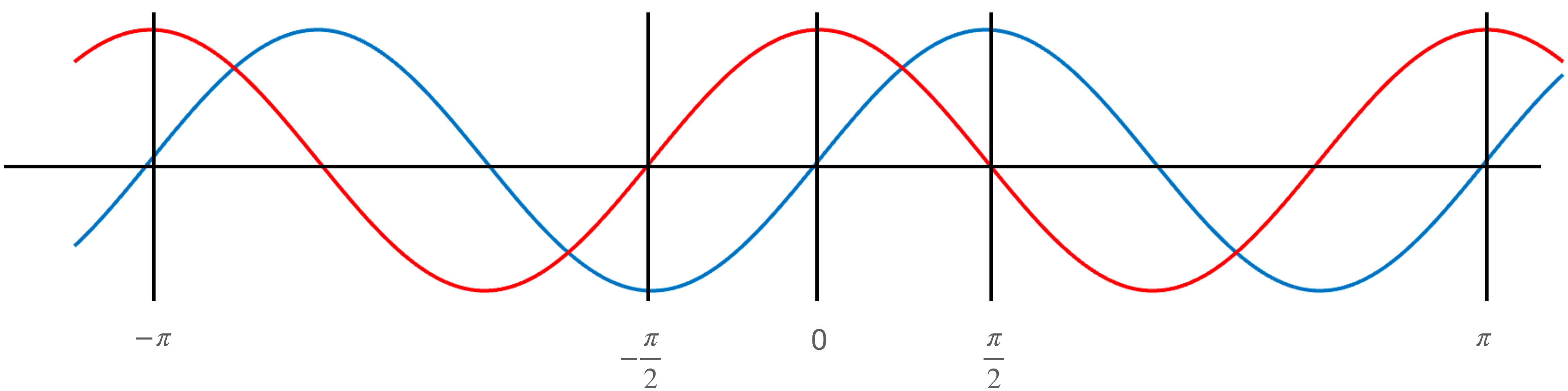

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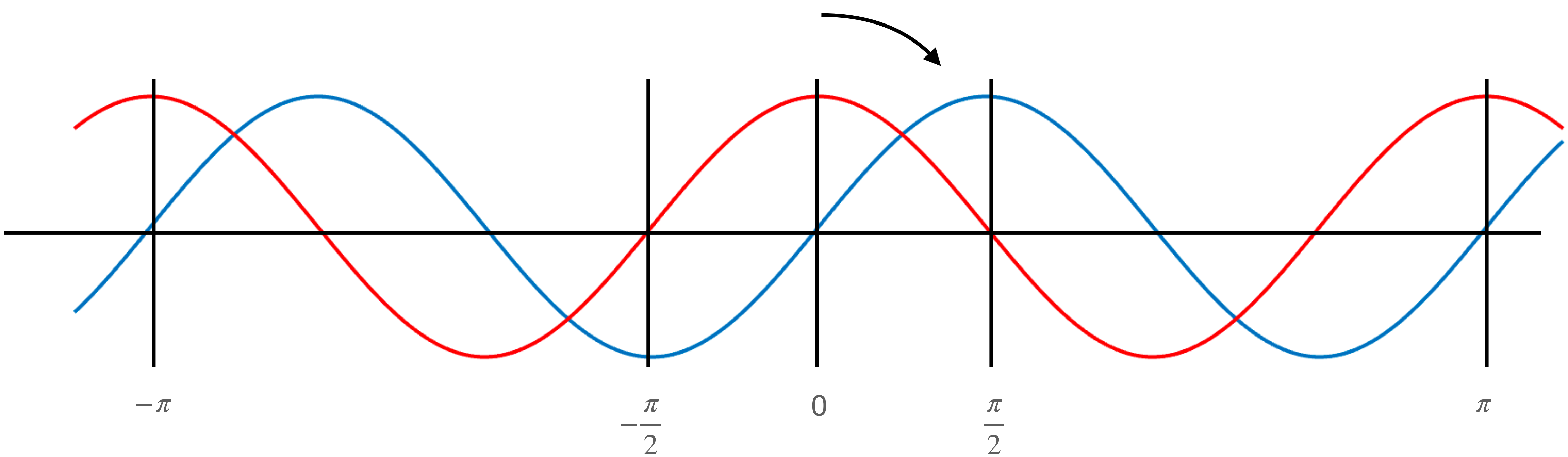

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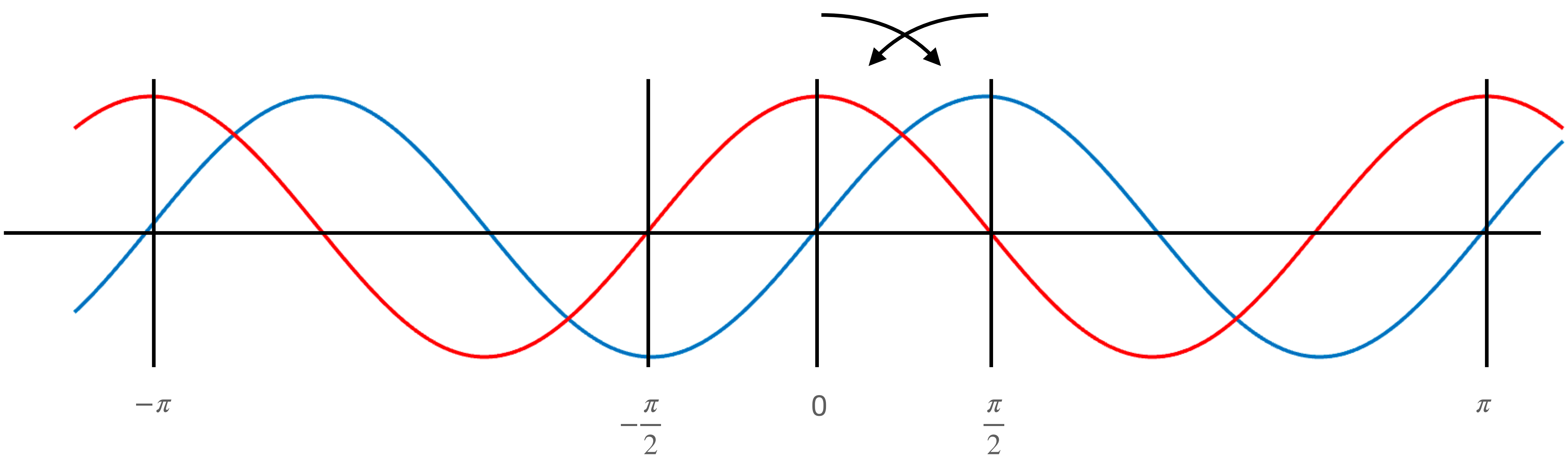

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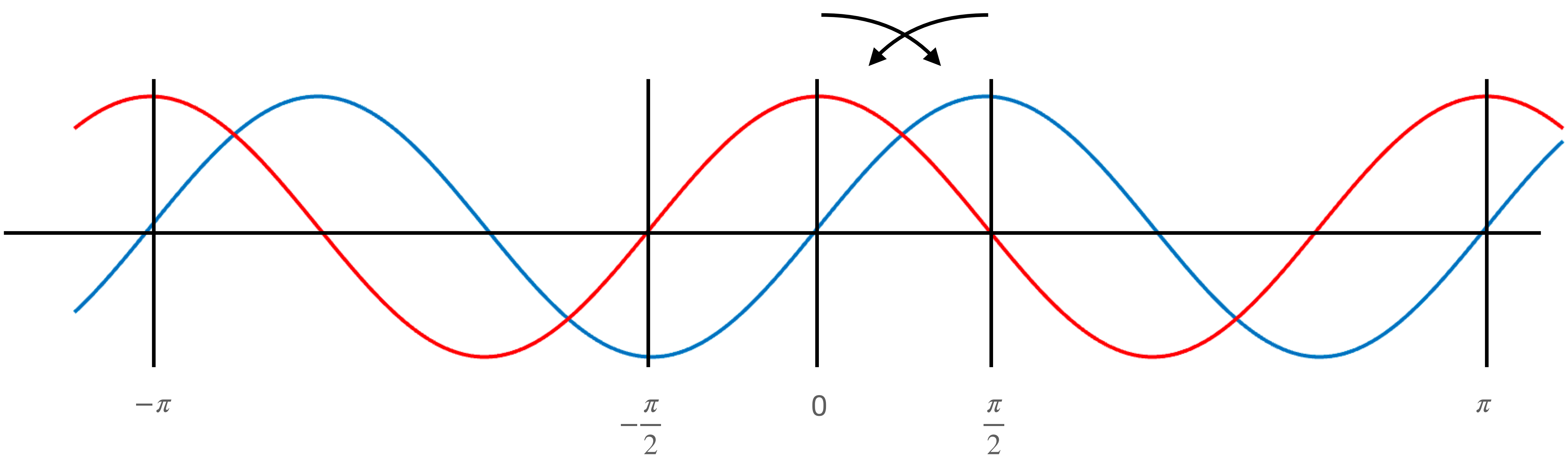

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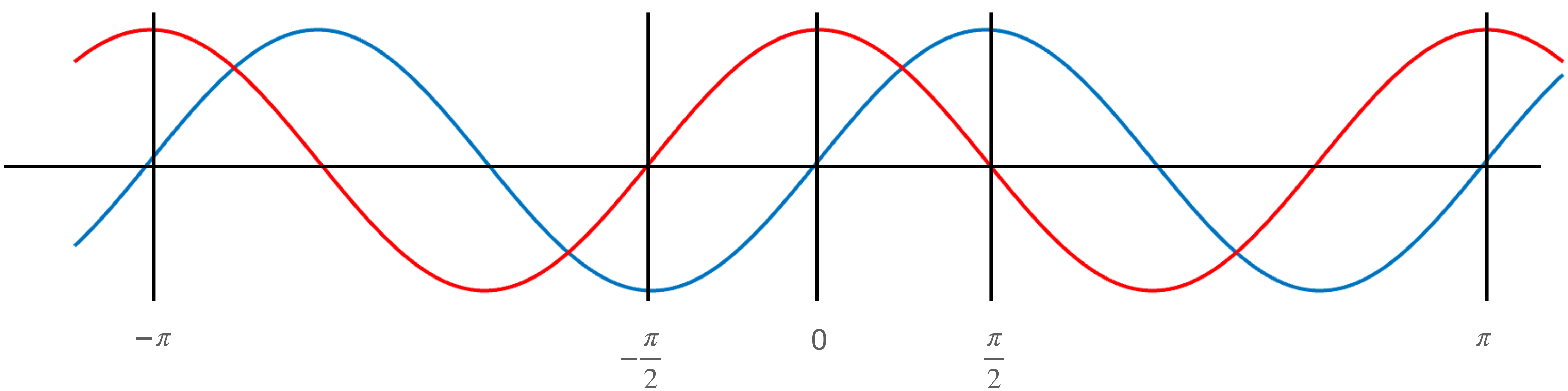

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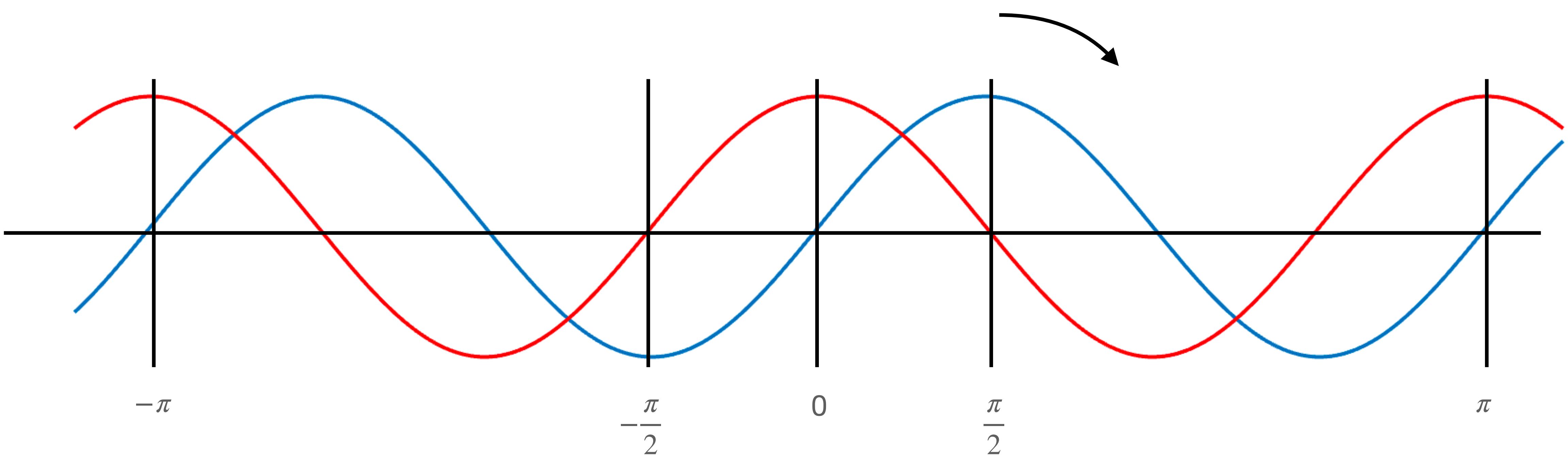

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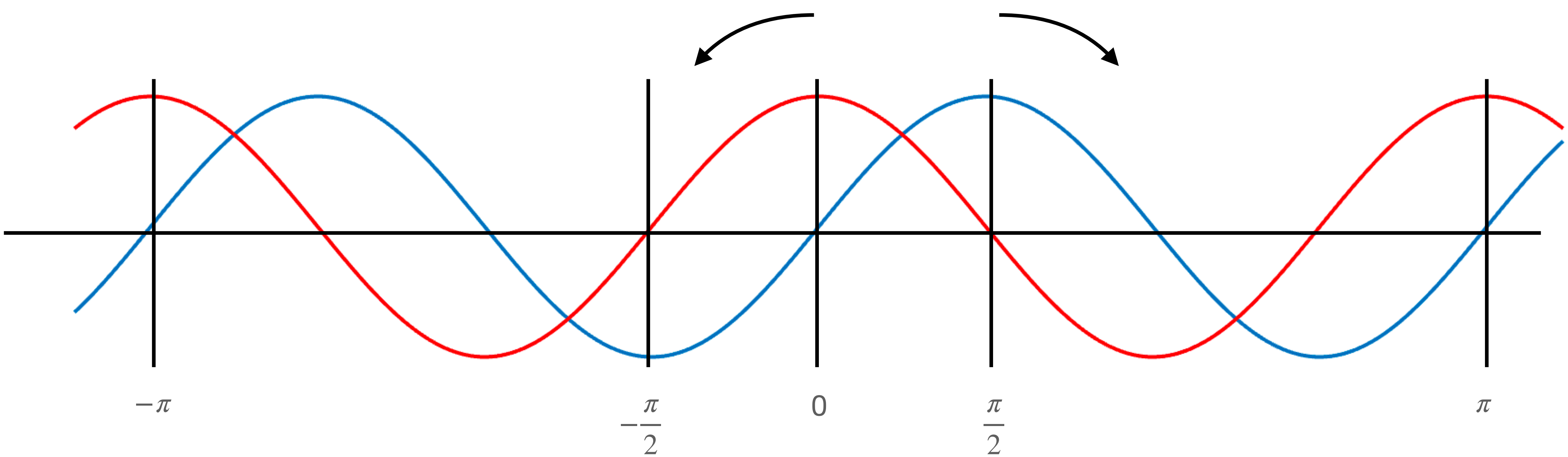

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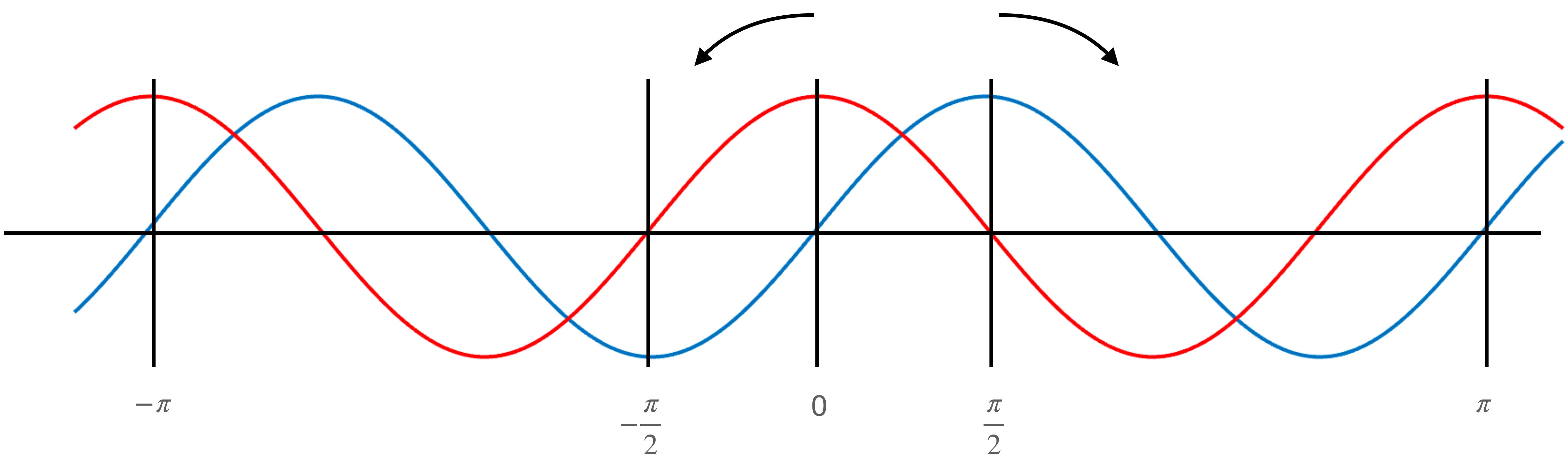

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And so…

With complex vectors in the imaginary plane:

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We can take (a,b), normalize,
work in unit vectors,
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$$sin(\beta - \alpha) = sin(\beta)cos(\alpha) - cos(\beta)sin(\alpha)$$

We can take (a,b), normalize,
work in unit vectors,
and multiply by needed length at the end

MULTIPLYING COMPLEX NUMBERS IS ROTATING

With complex vectors in the imaginary plane:

Multiplying vectors is adding angles

$$cos(\beta + \alpha) = cos(\beta)cos(\alpha) - sin(\beta)sin(\alpha)$$

$$sin(\beta + \alpha) = sin(\beta)cos(\alpha) + cos(\beta)sin(\alpha)$$

Q0:

$$cos(\beta - \alpha) = cos(\beta)cos(\alpha)) + sin(\beta)sin(\alpha)$$

$$sin(\beta - \alpha) = sin(\beta)cos(\alpha) - cos(\beta)sin(\alpha)$$

We can take (a,b), normalize,
work in unit vectors,
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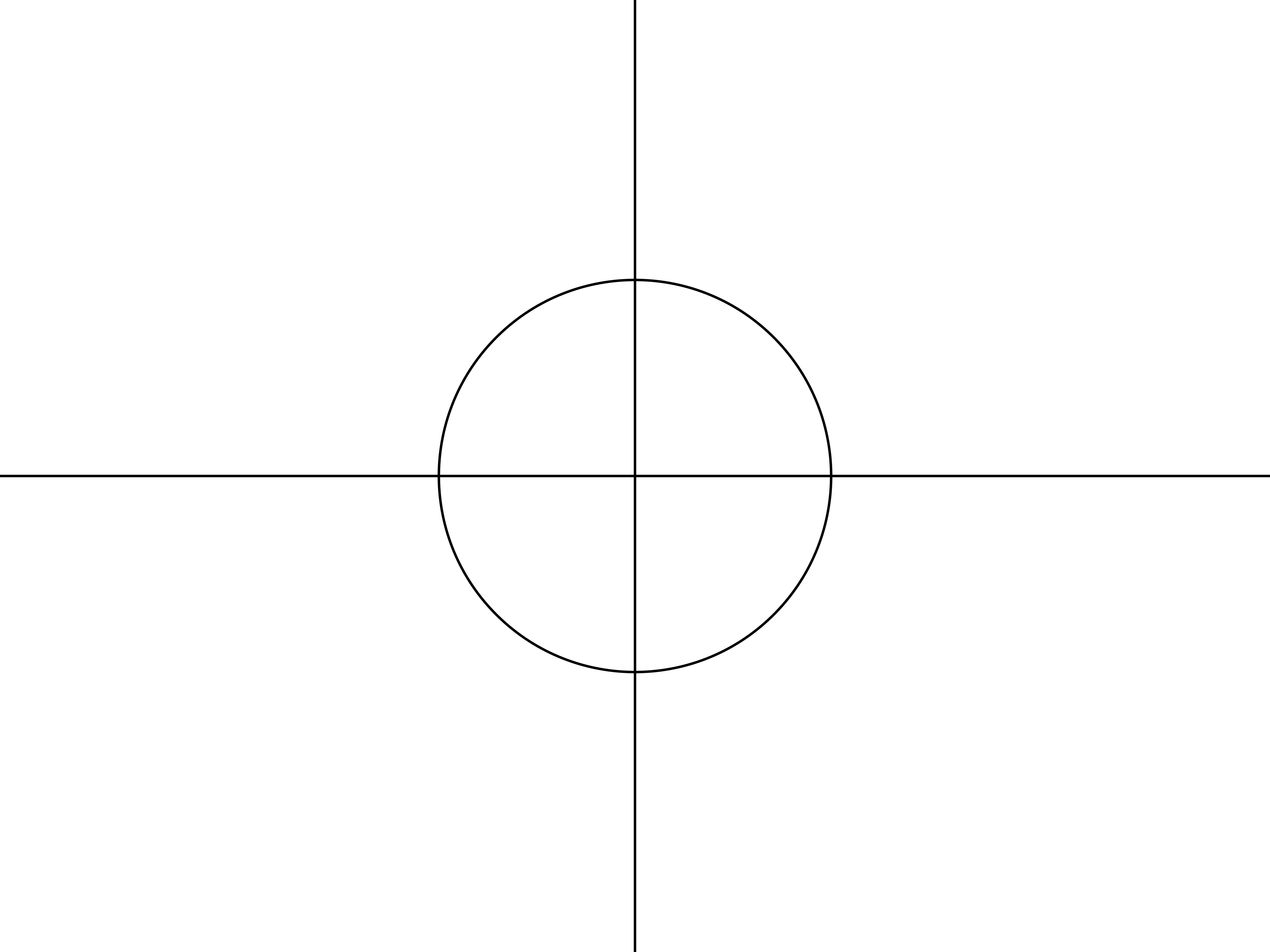
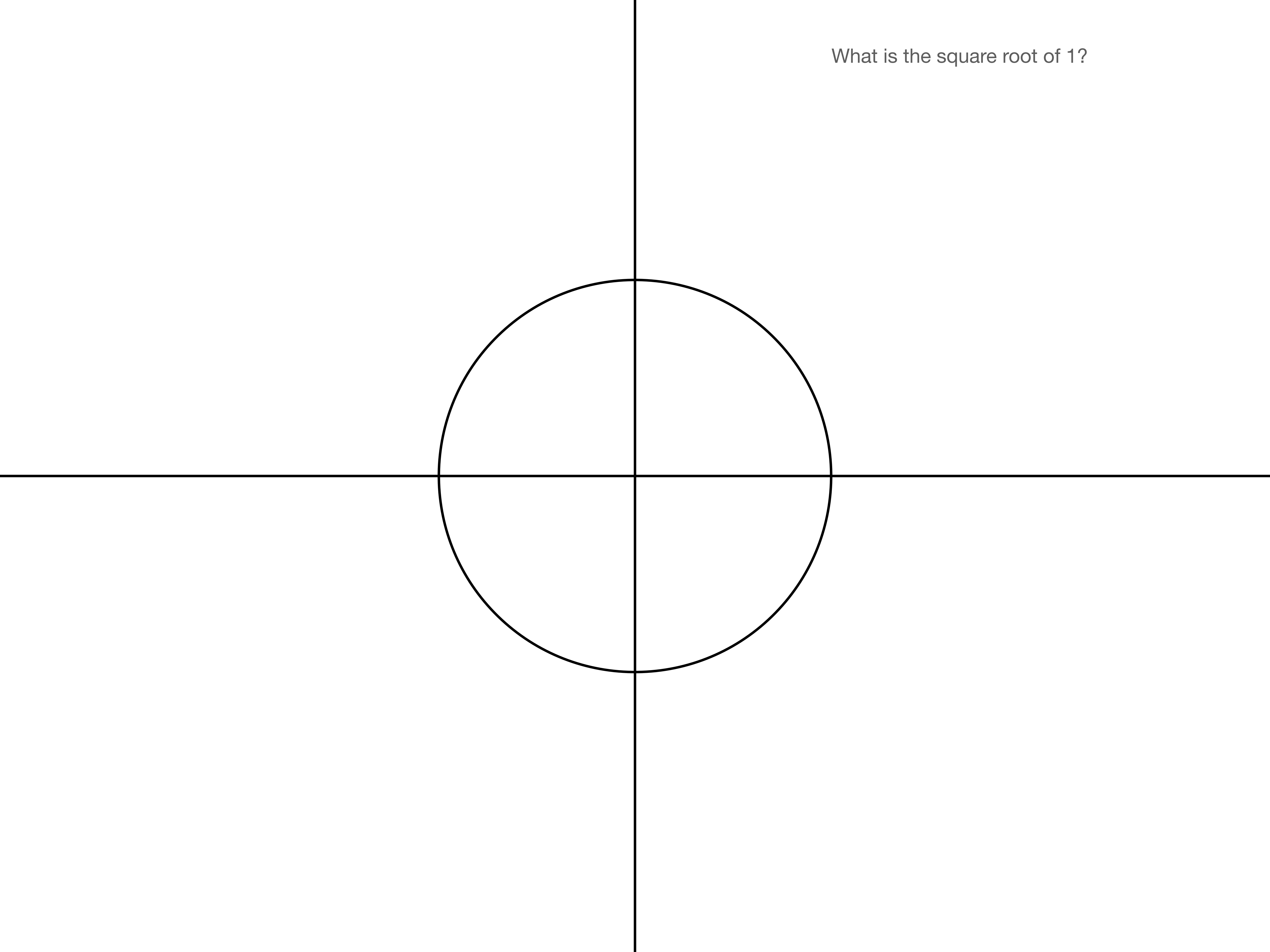
MULTIPLYING COMPLEX NUMBERS IS ROTATING



What is the square root of 1?



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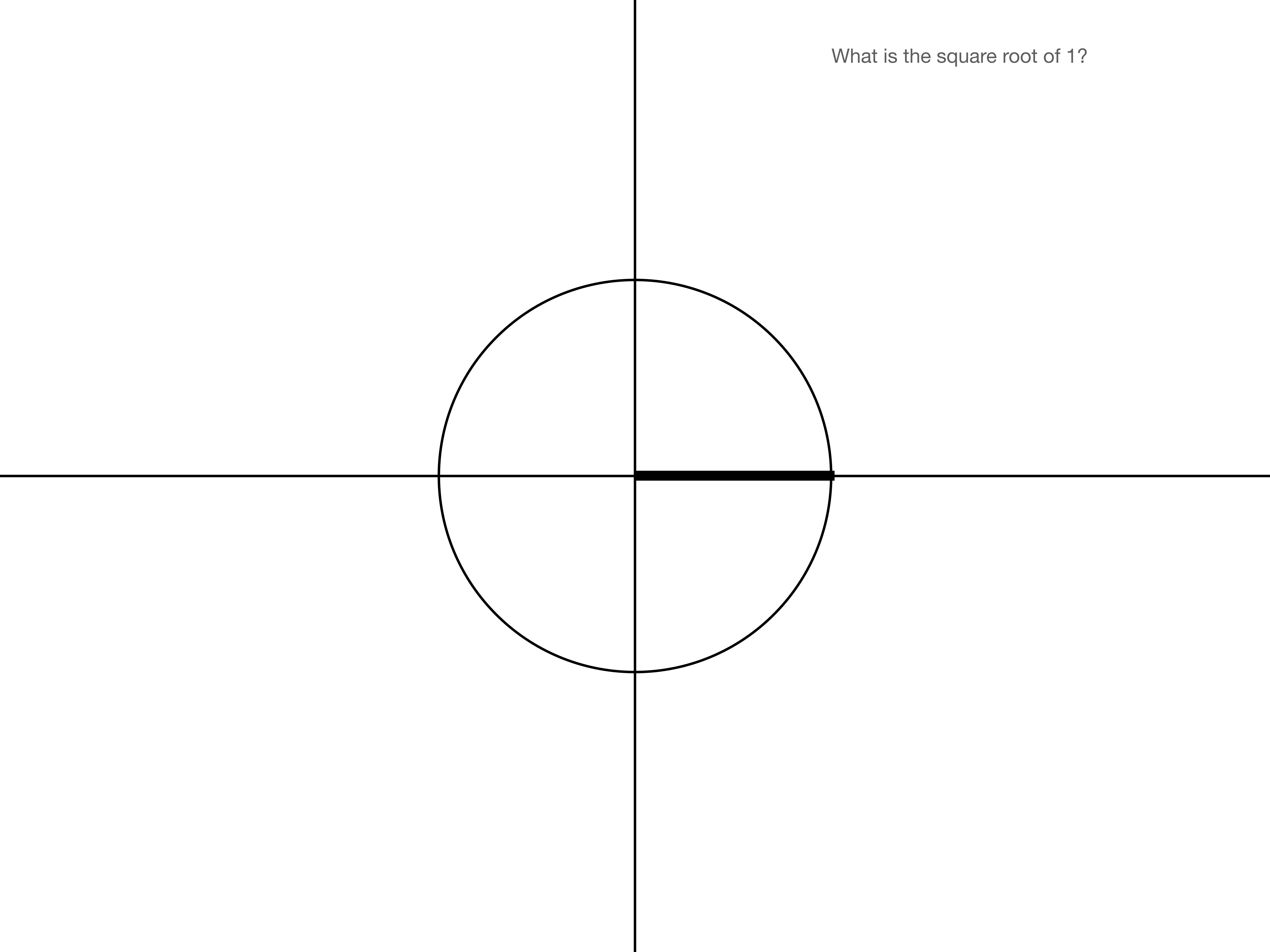

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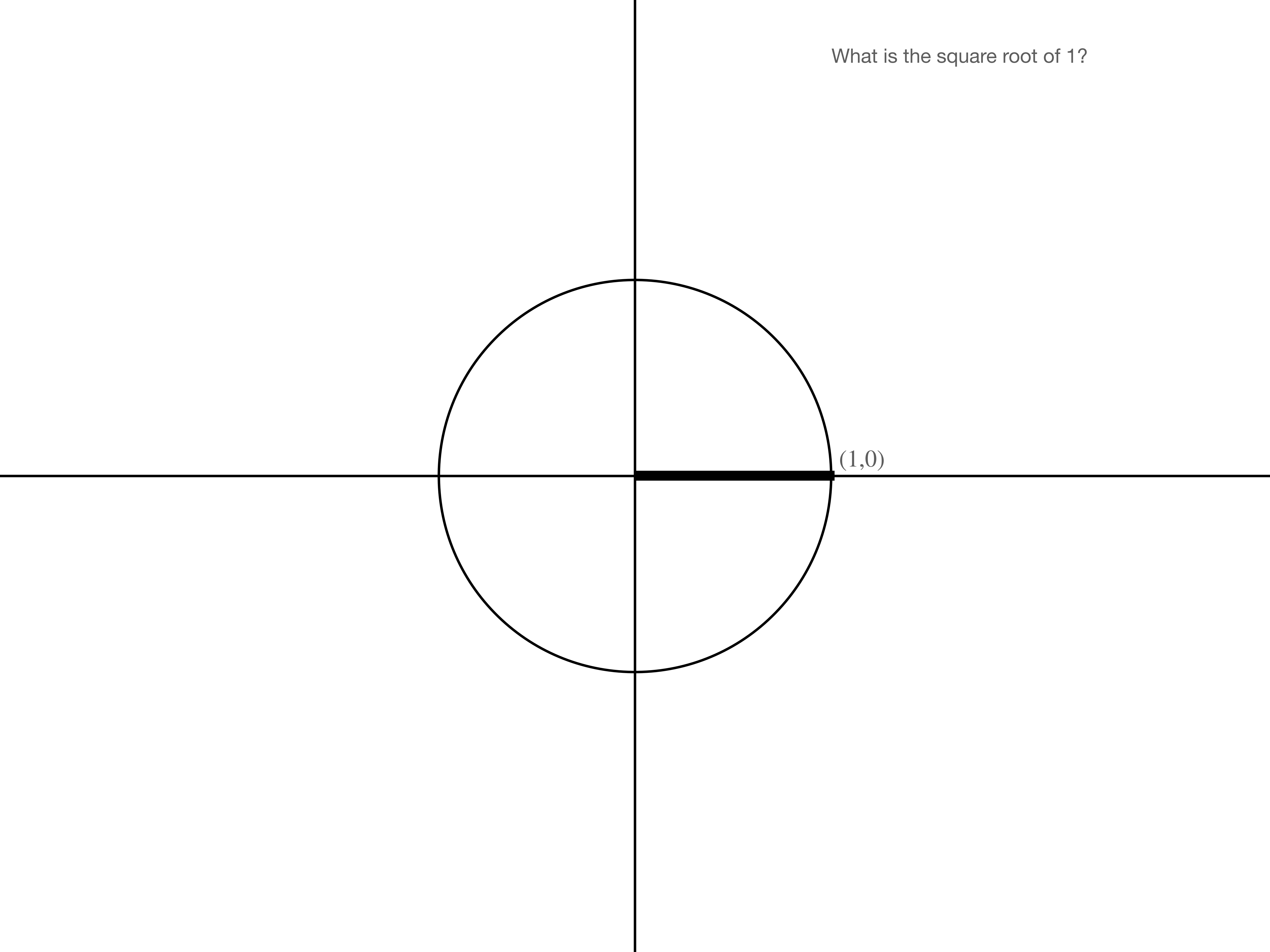
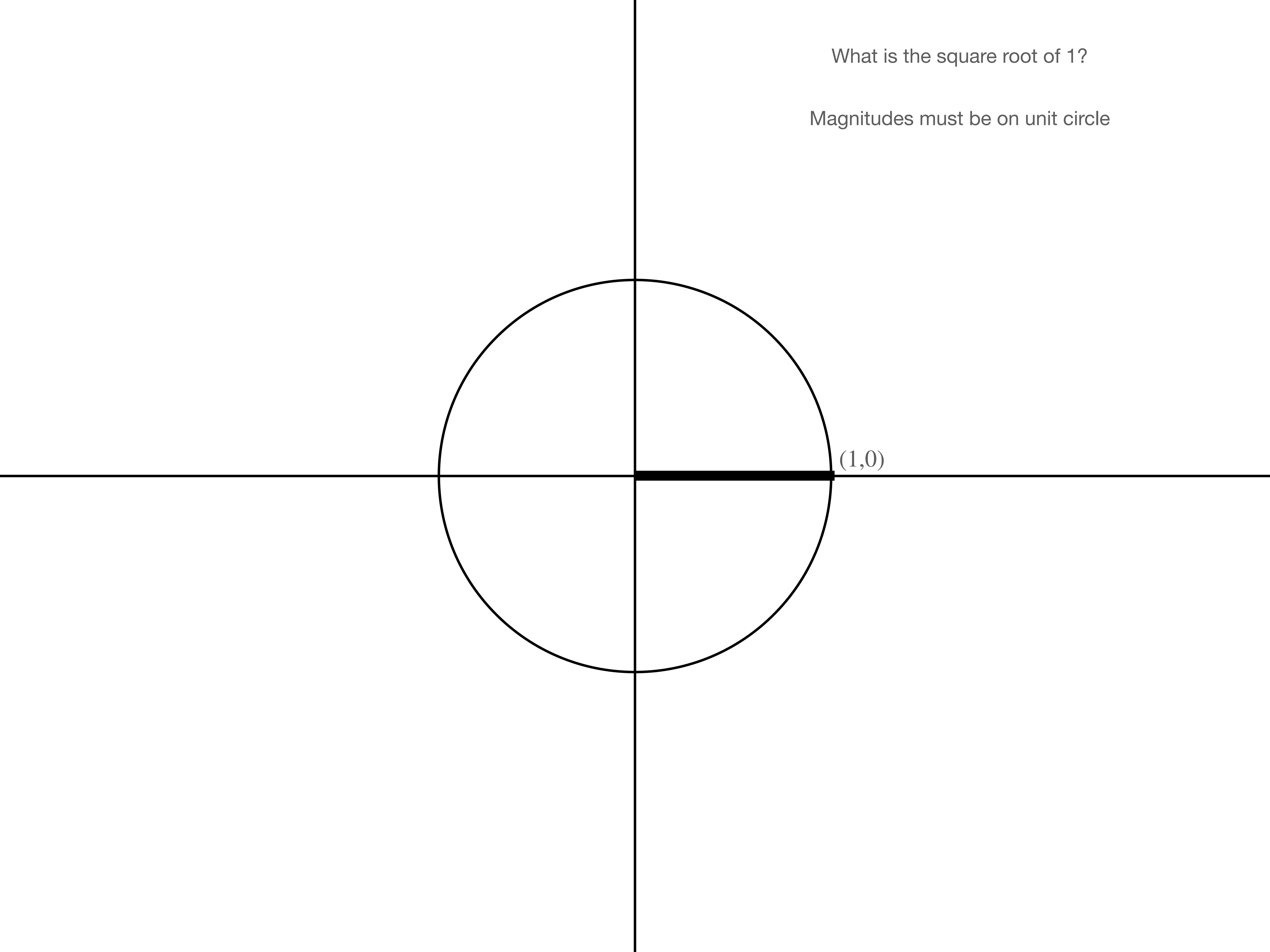


What is the square root of 1?

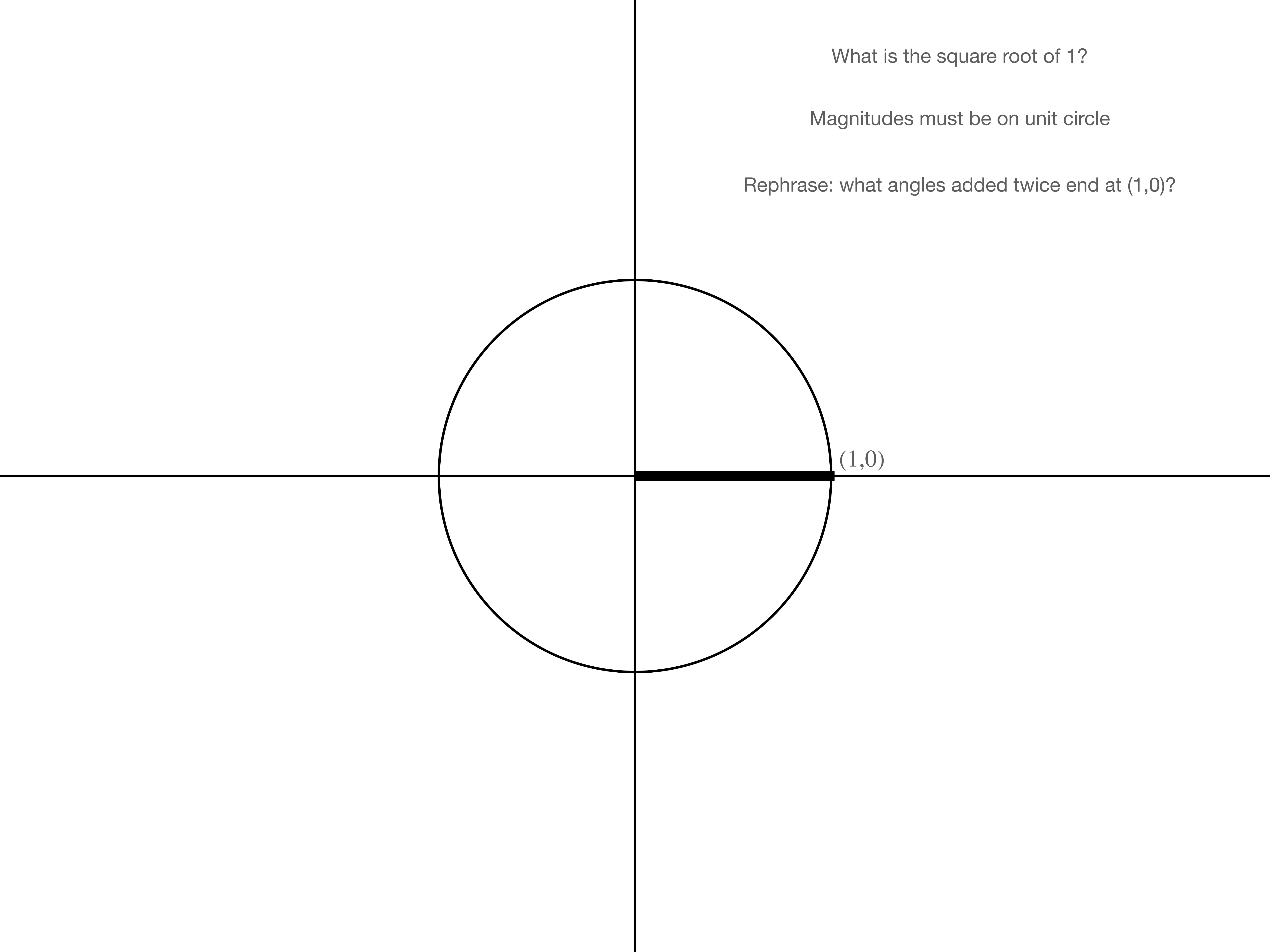
Magnitudes must be on unit circle



What is the square root of 1?

Magnitudes must be on unit circle

Rephrase: what angles added twice end at (1,0)?



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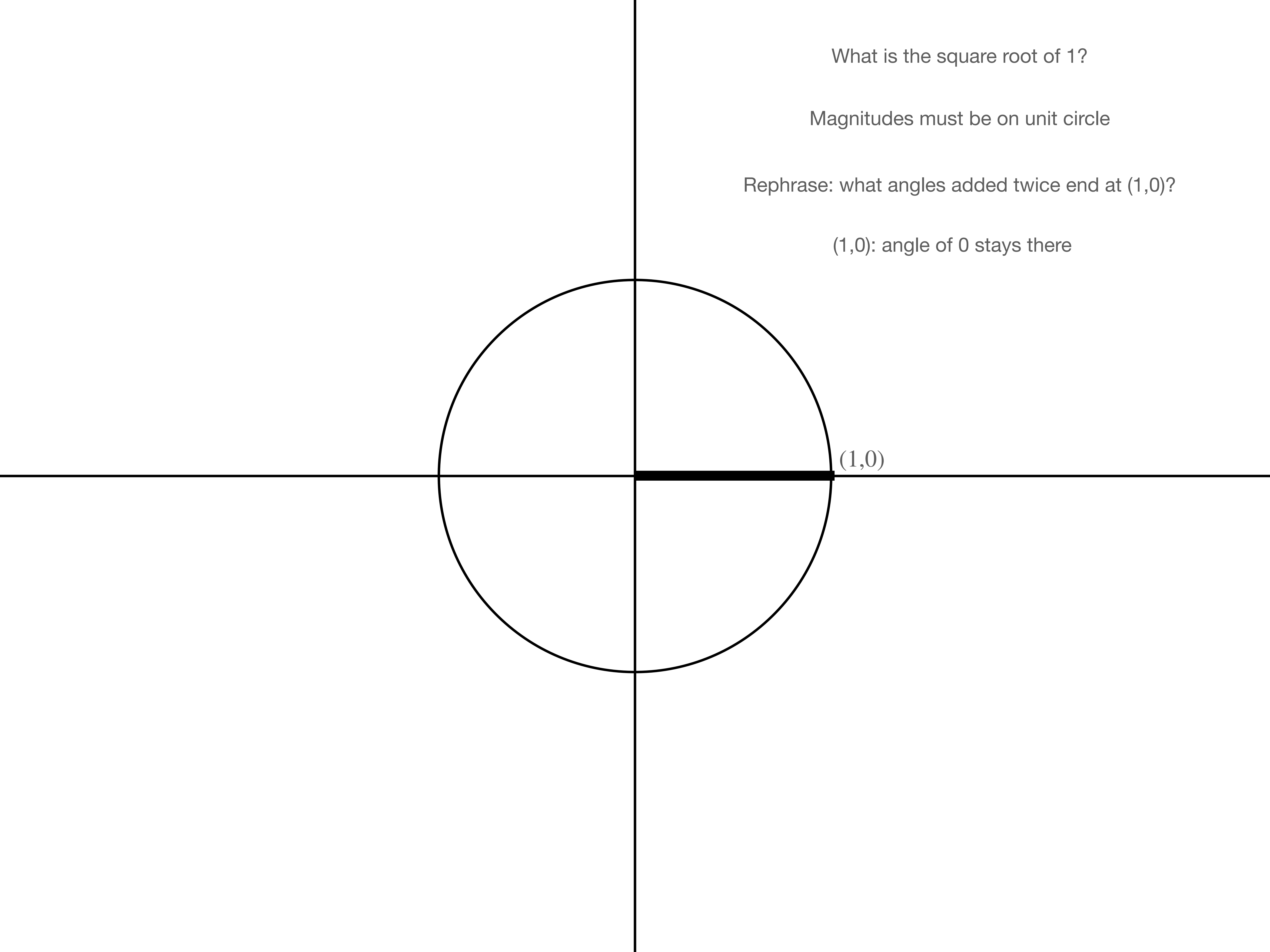
Rephrase: what angles added twice end at (1,0)?

(1,0): angle of 0 stays there



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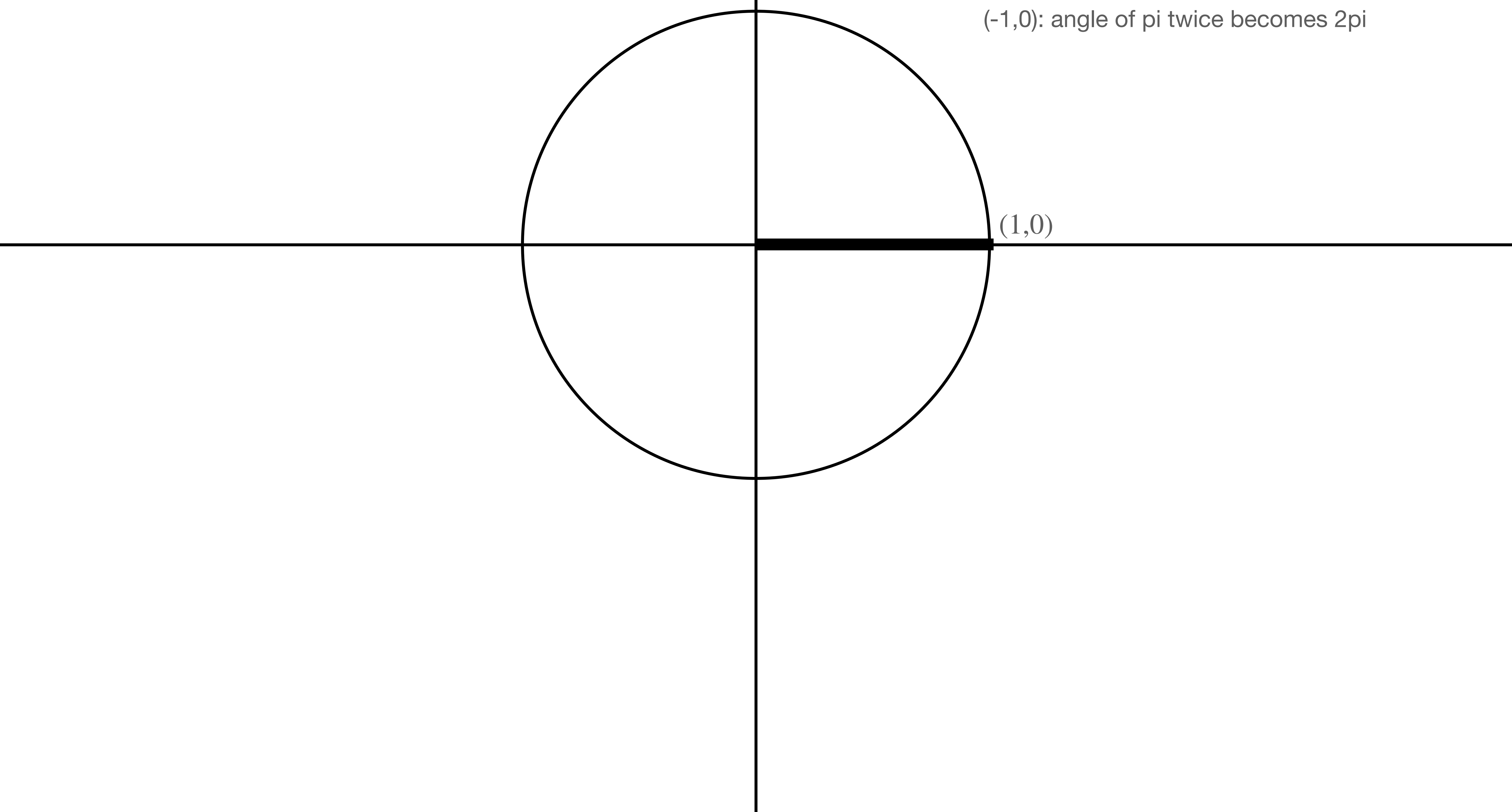
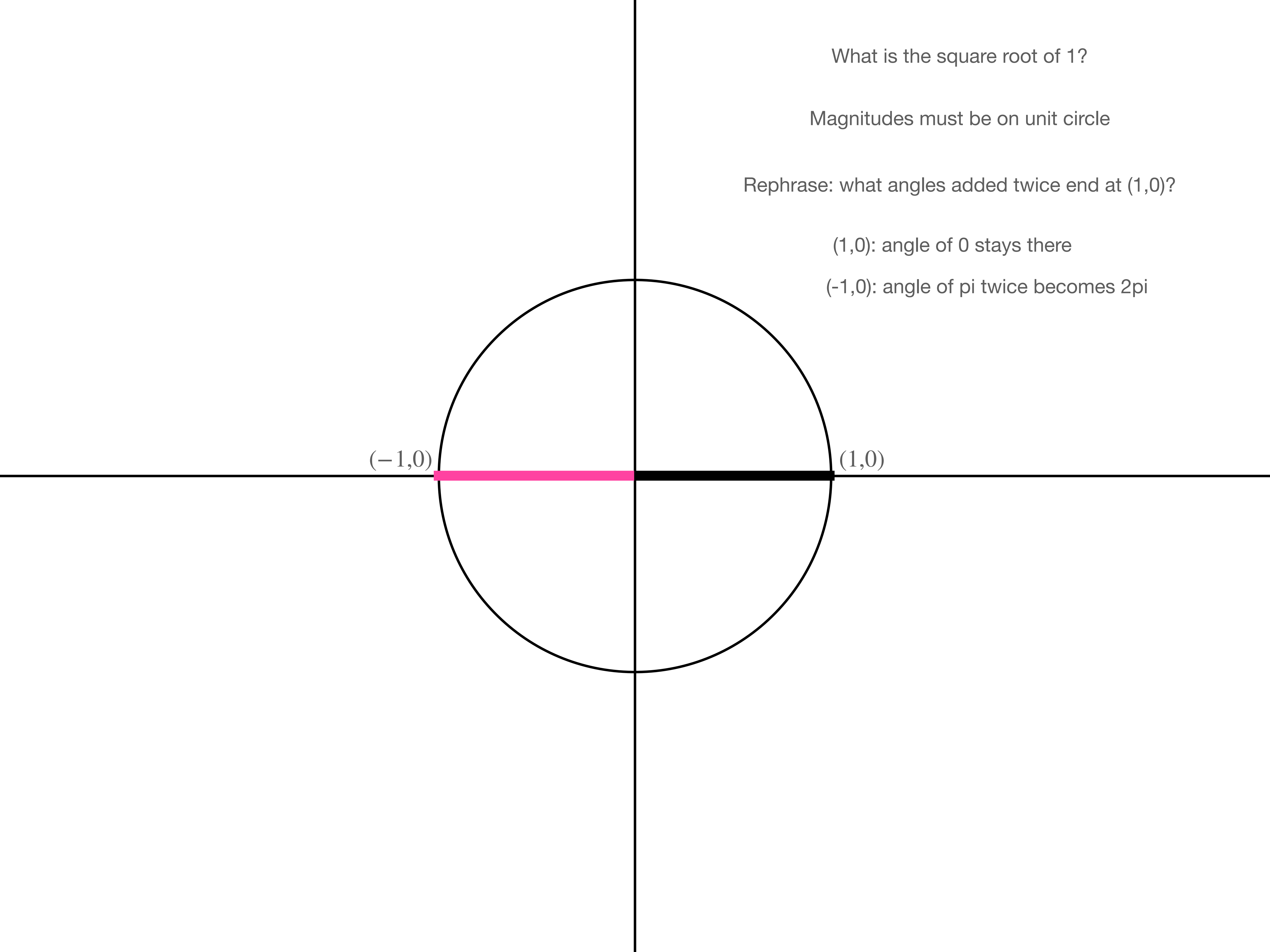
Rephrase: what angles added twice end at (1,0)?

(1,0): angle of 0 stays there

(-1,0): angle of pi twice becomes 2pi





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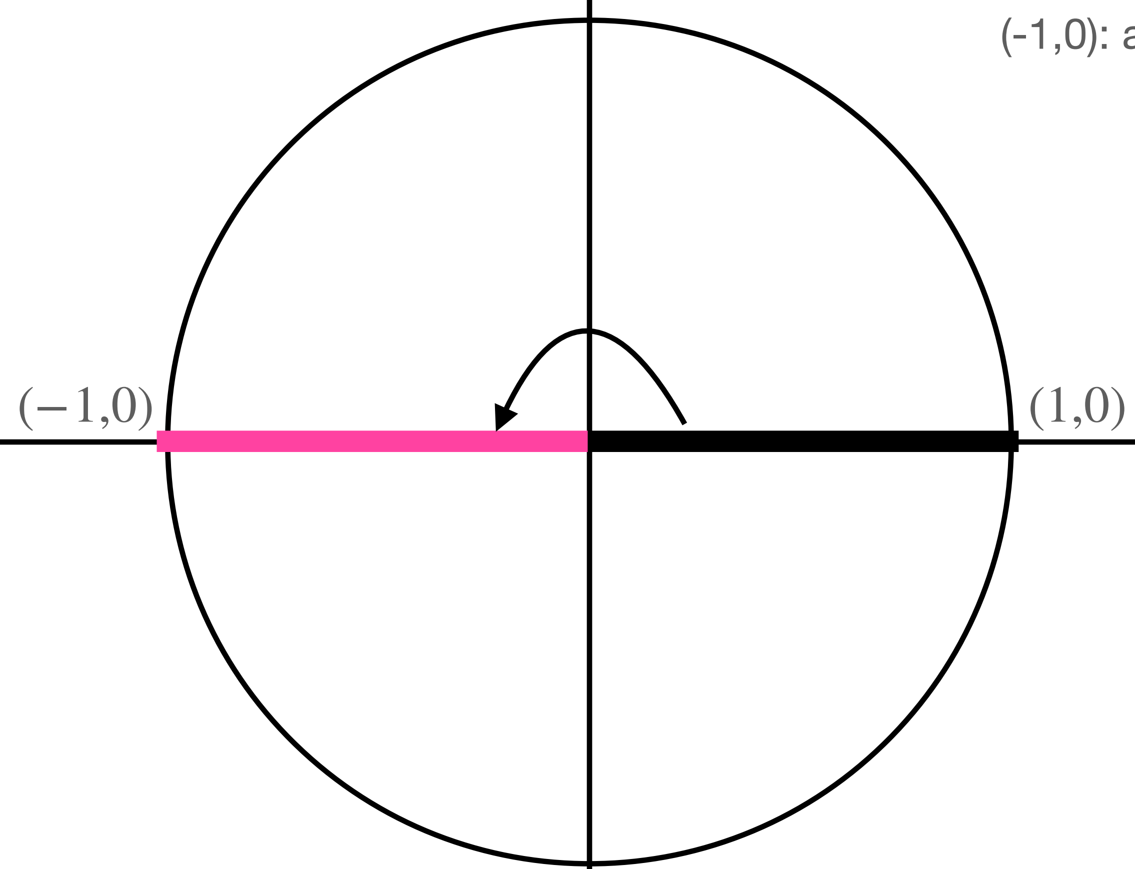

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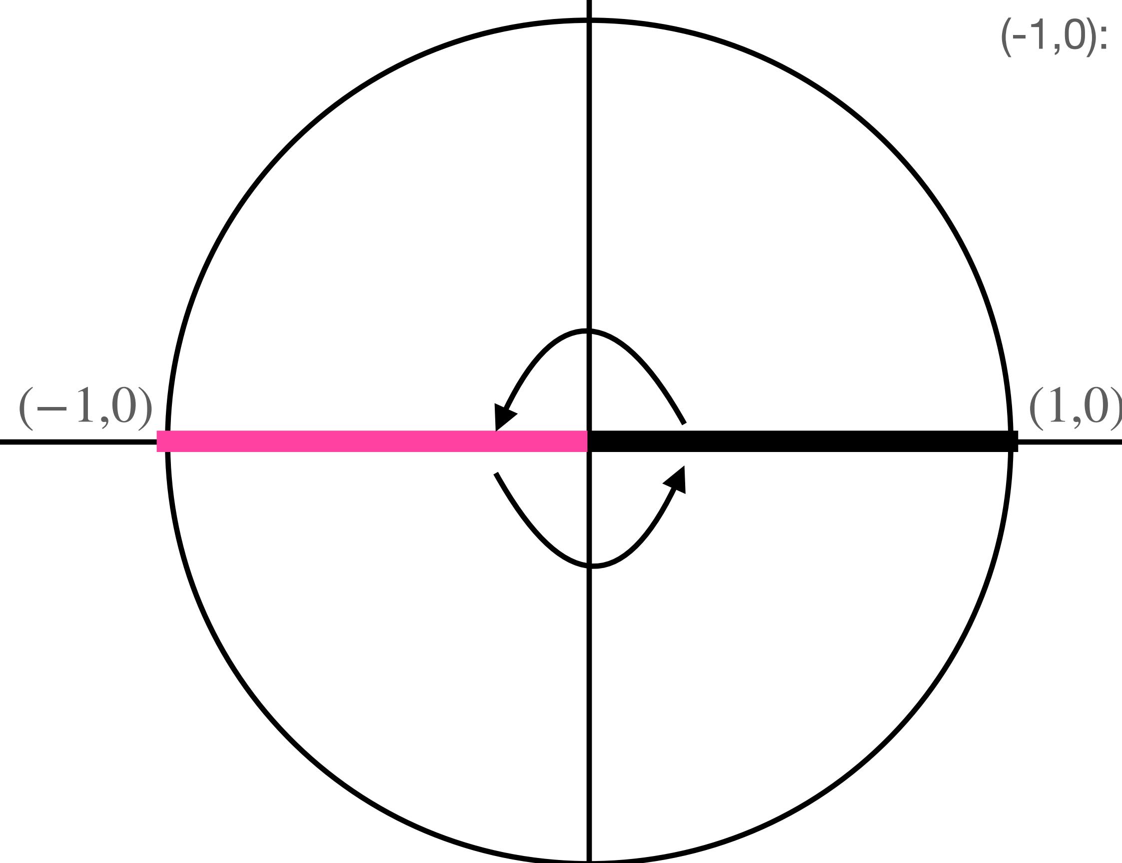
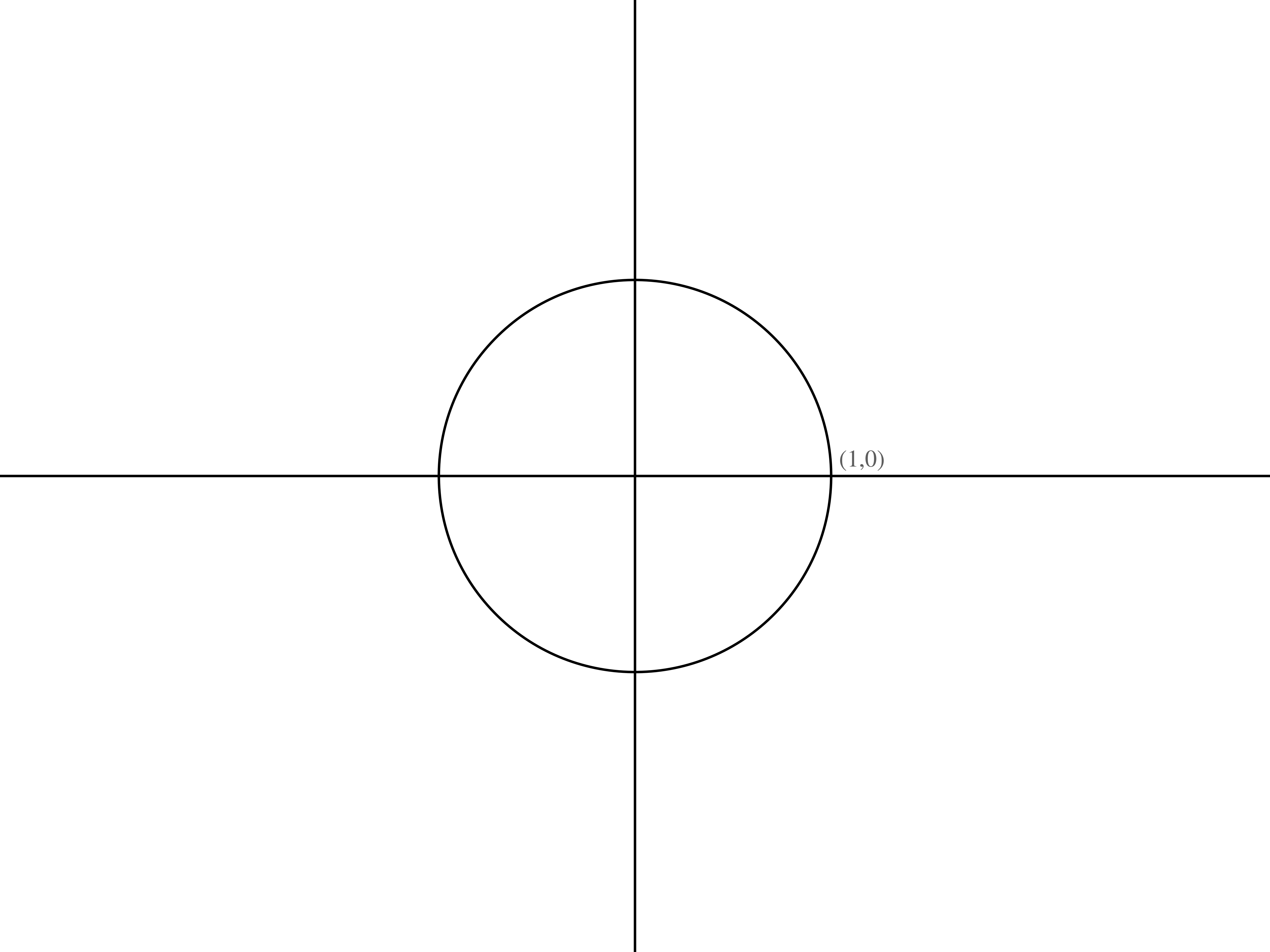
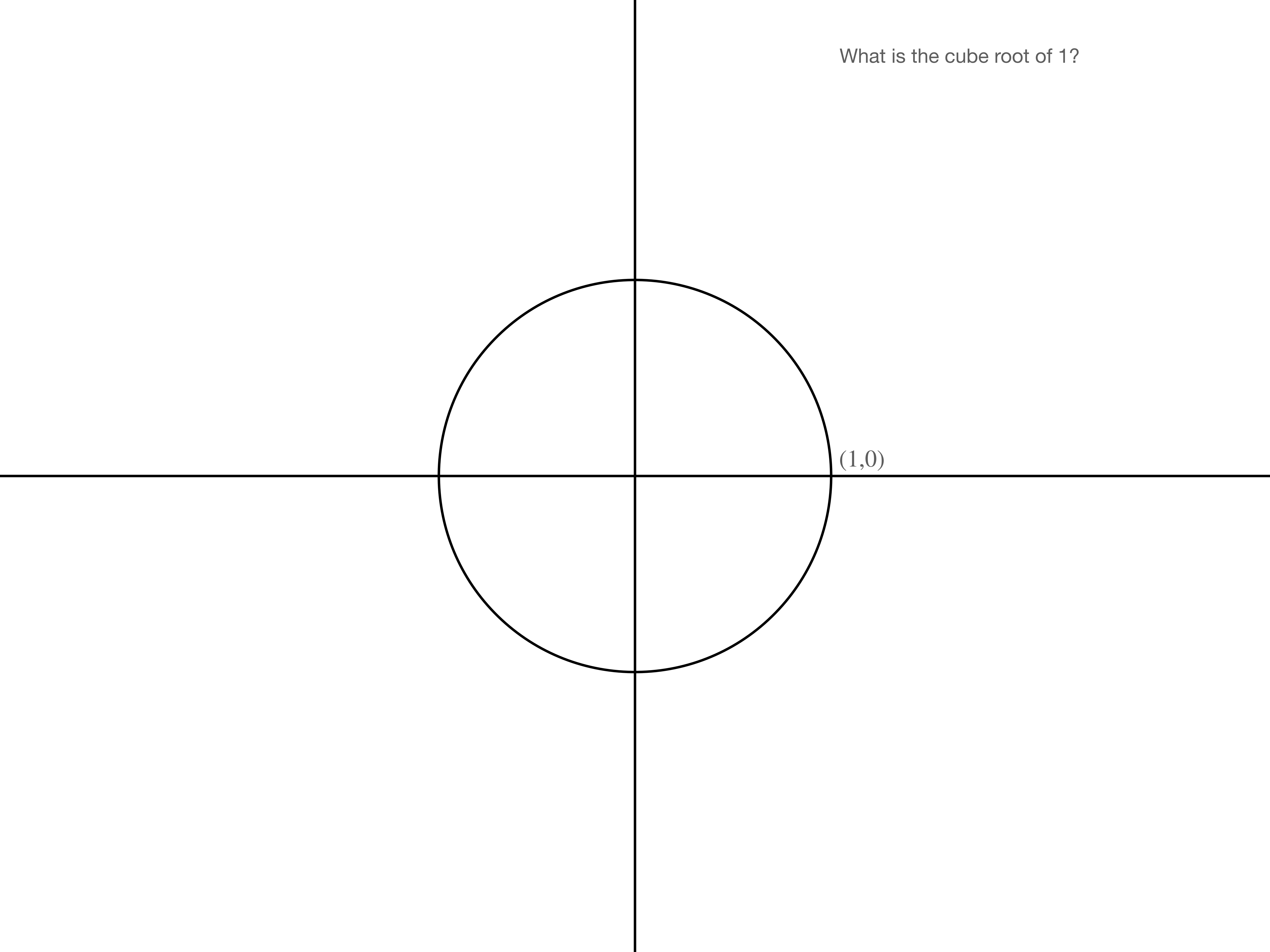


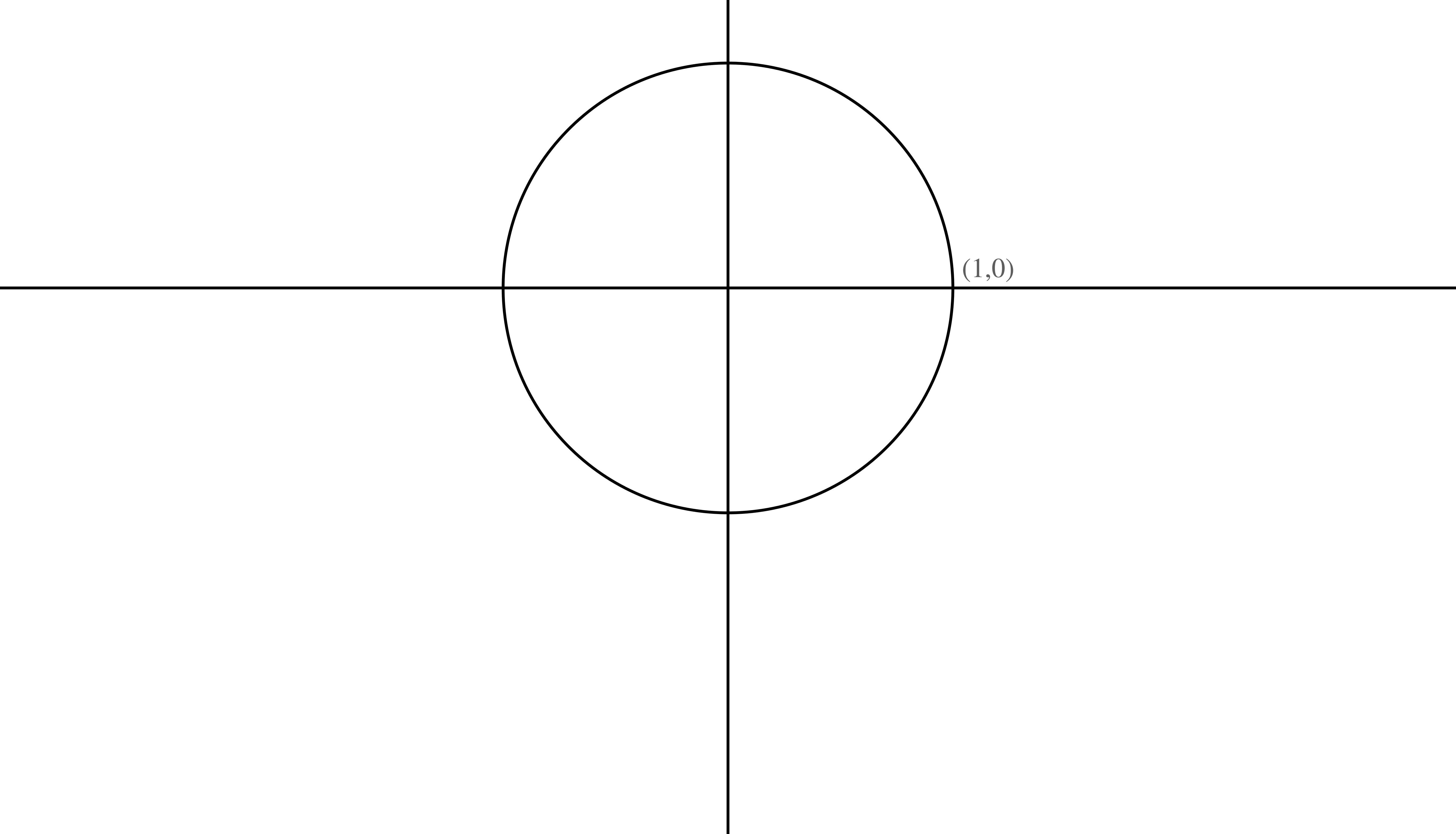

(1,0)

What is the cube root of 1?



What is the cube root of 1?

Rephrase: what angles added three times are multiples of 2pi?



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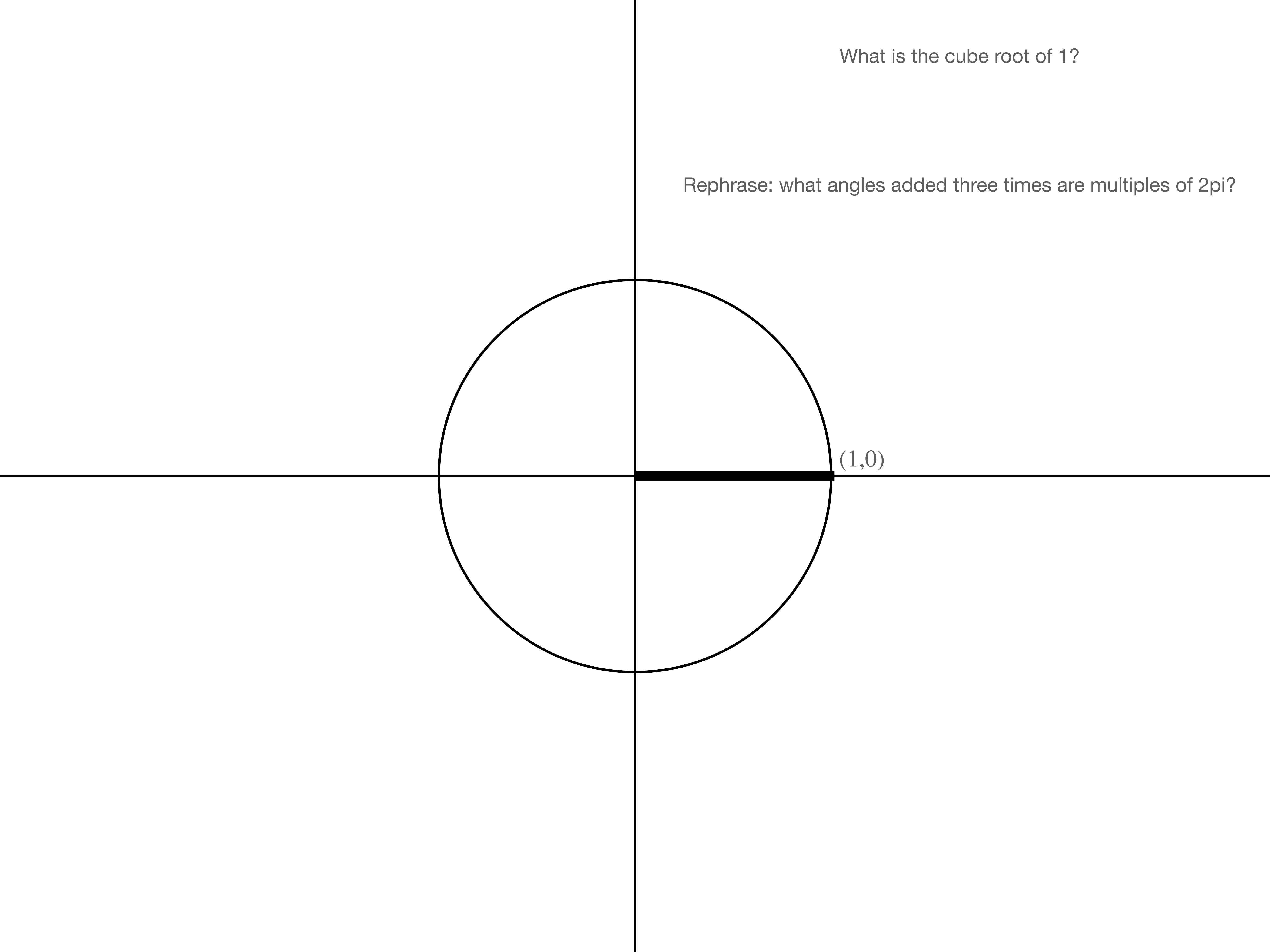

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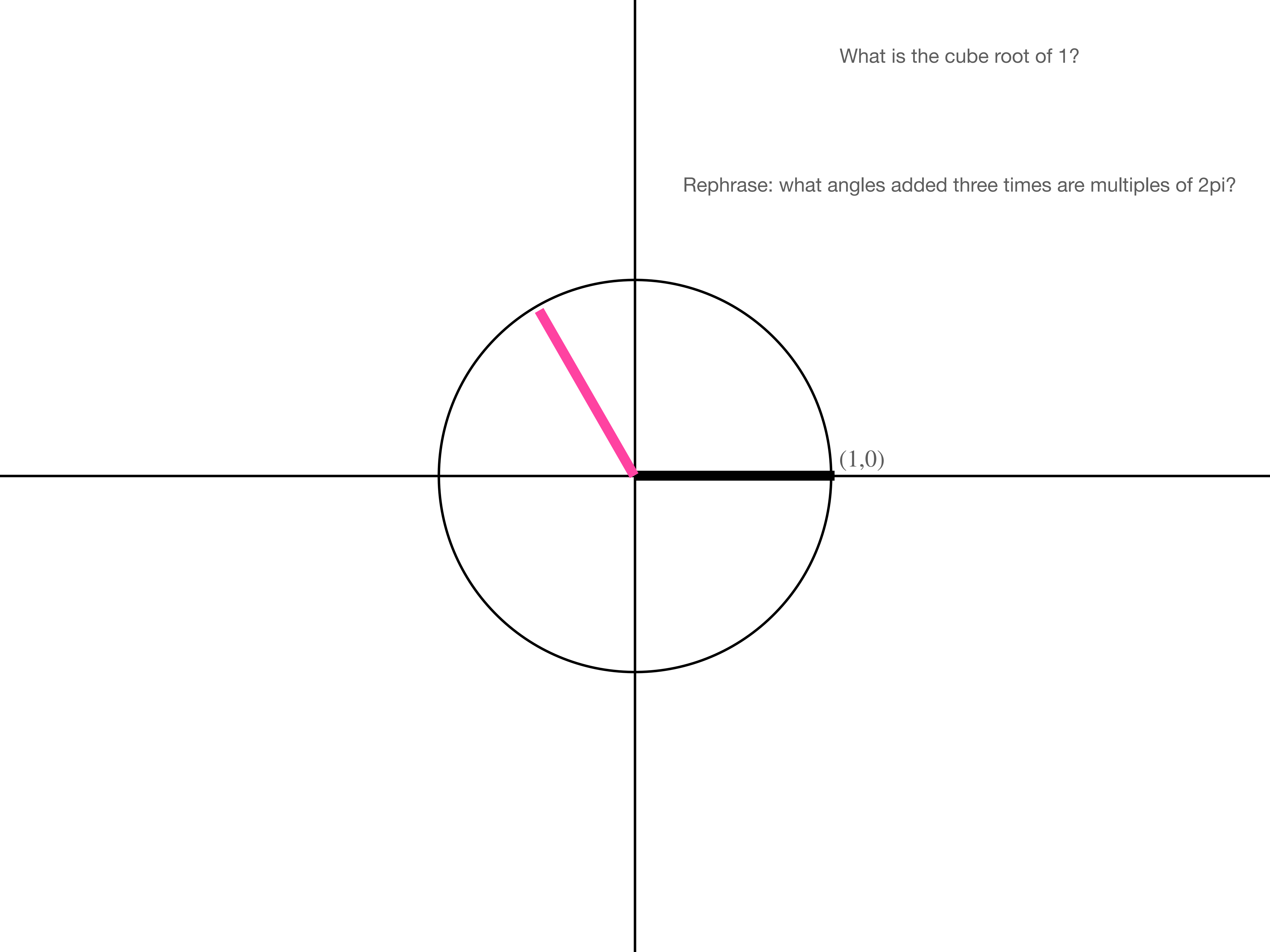

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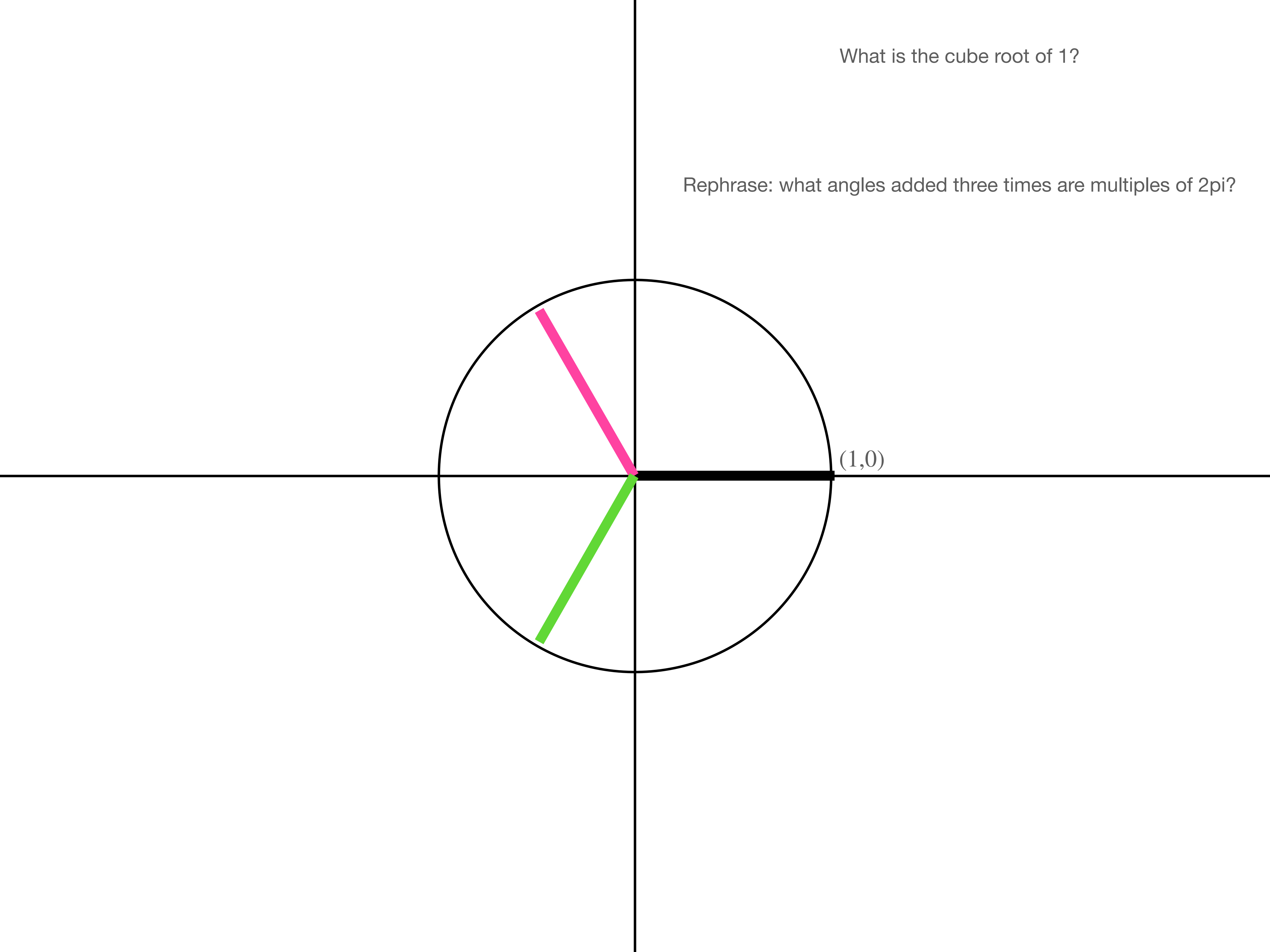
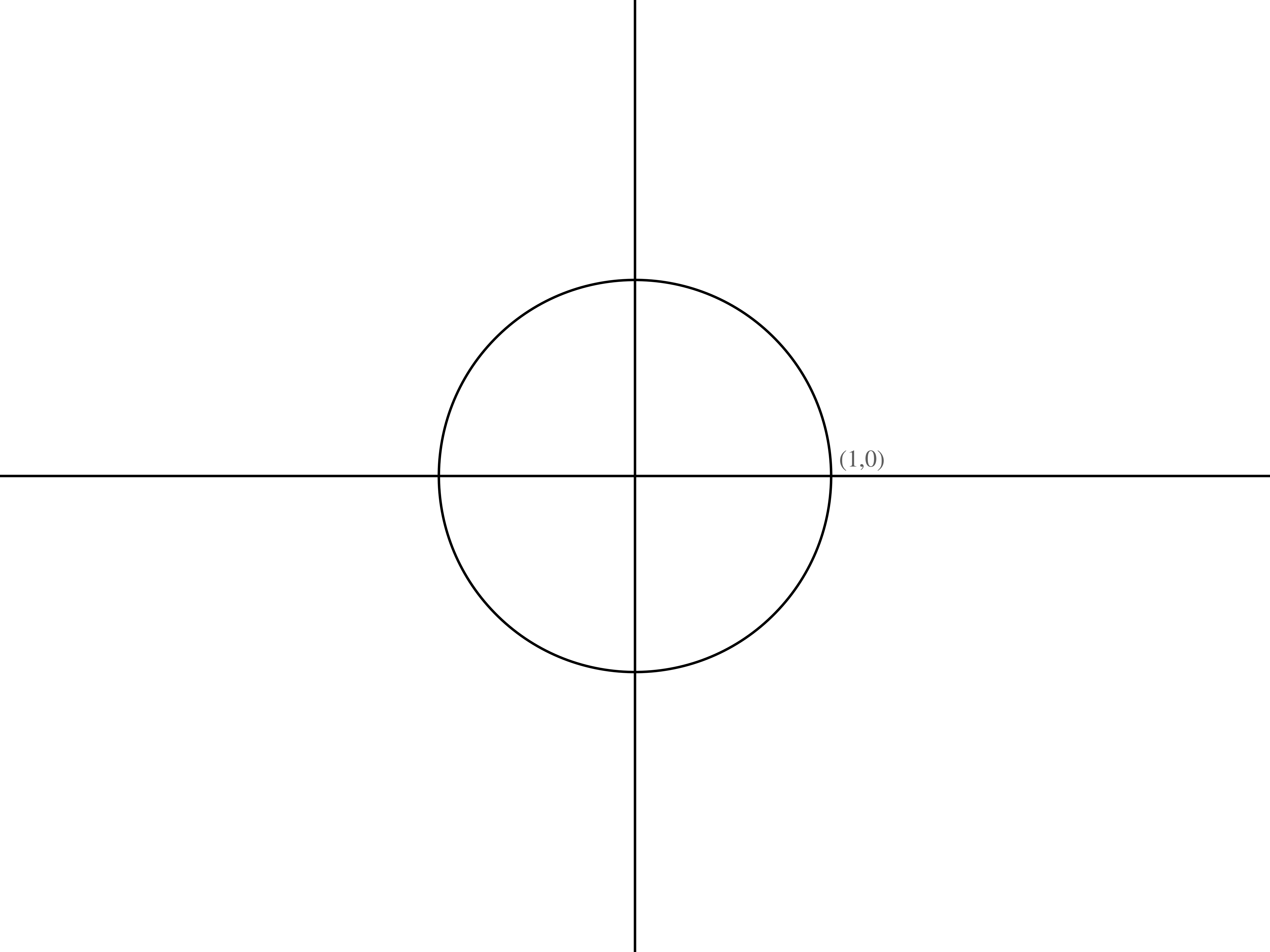
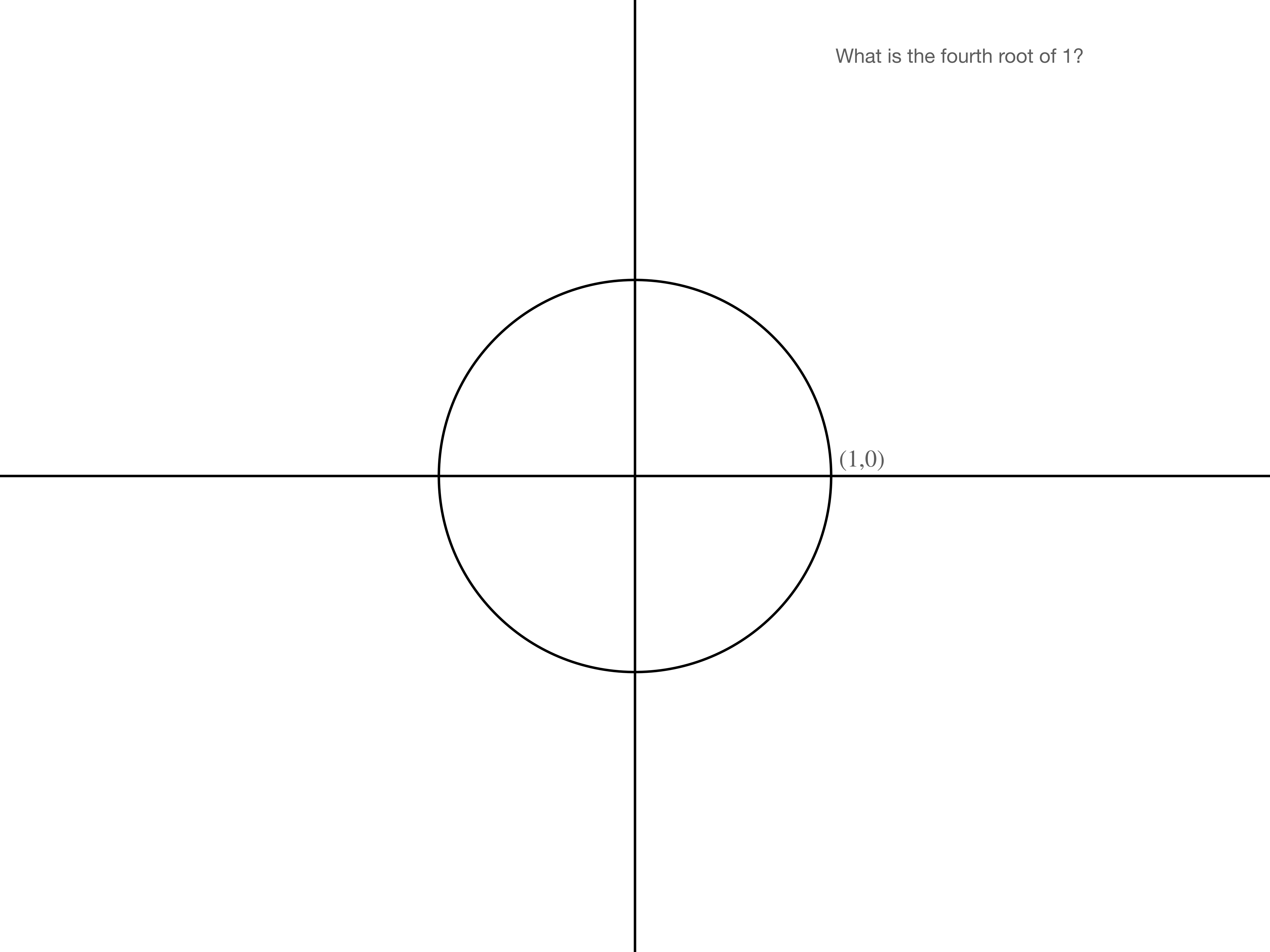


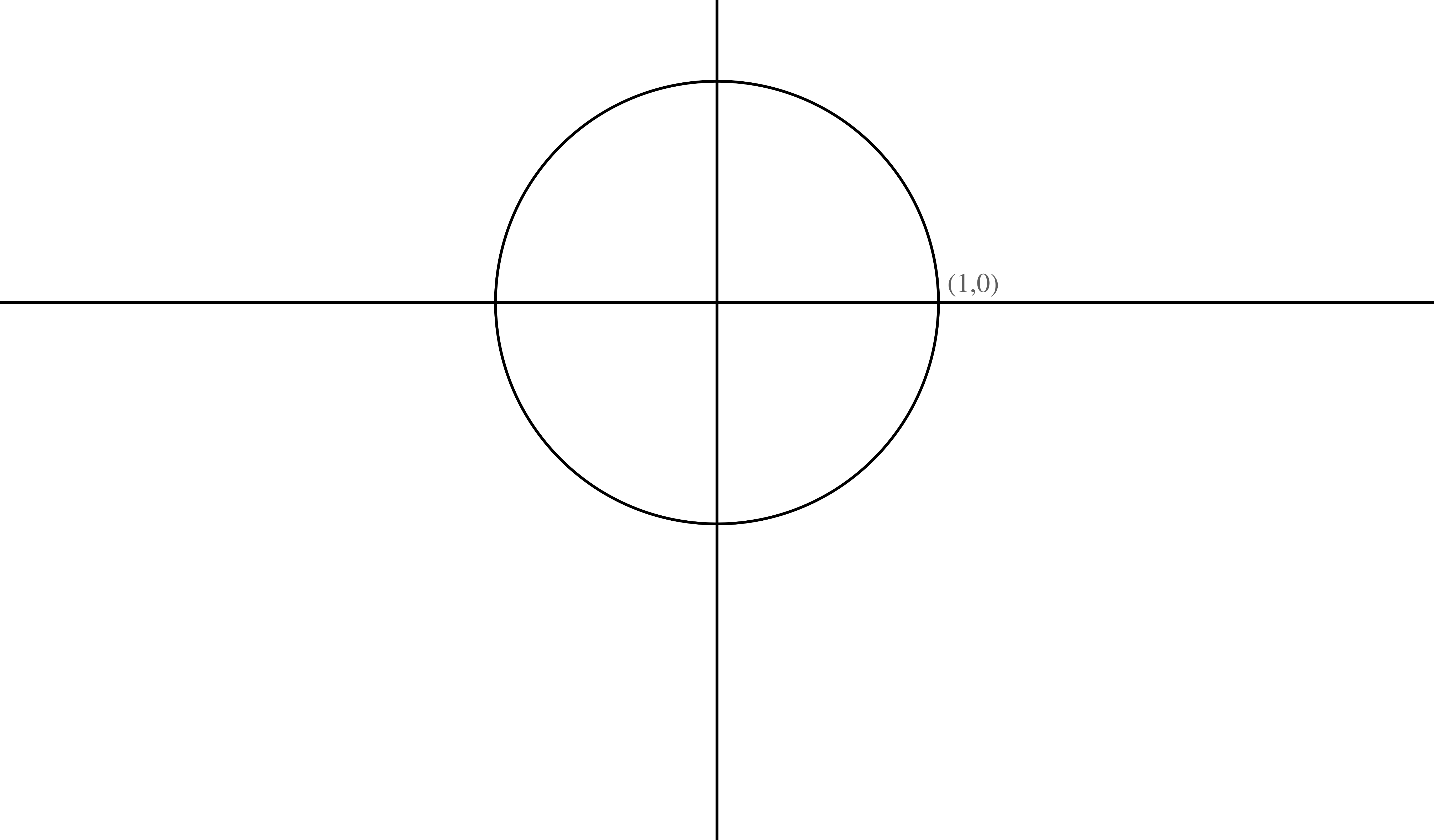

(1,0)

What is the fourth root of 1?



What is the fourth root of 1?

Rephrase: what angles added four times are 0?



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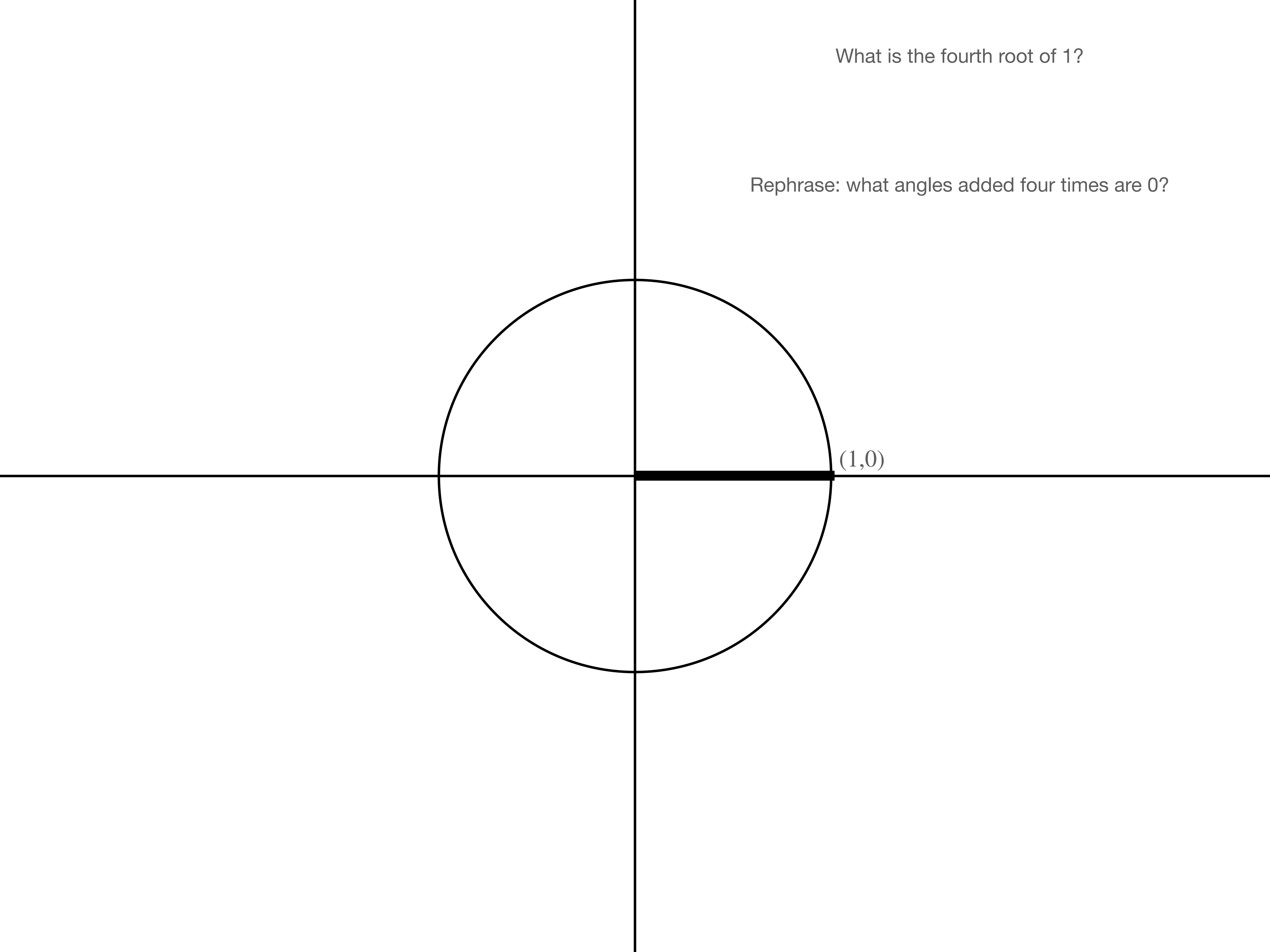

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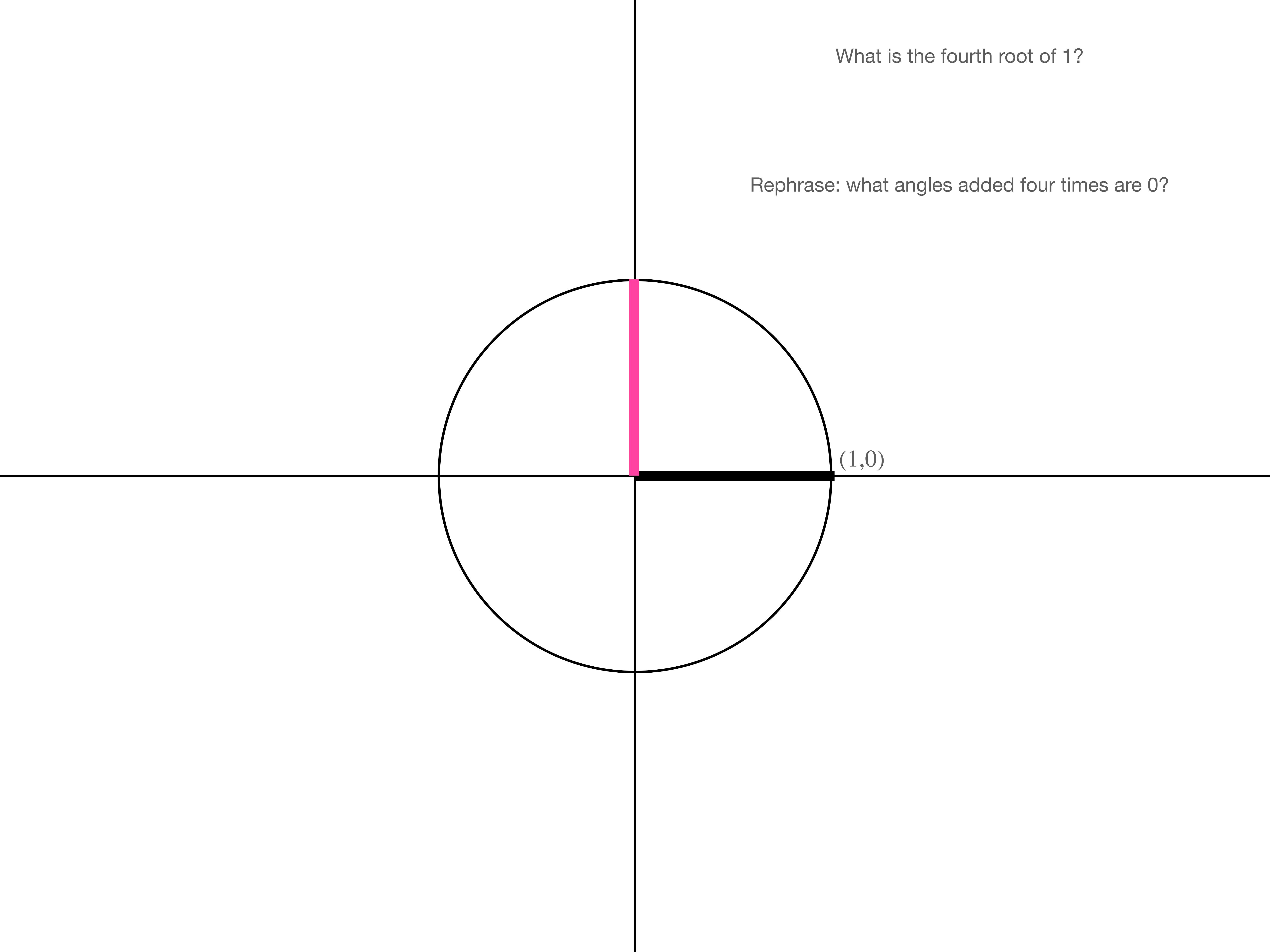

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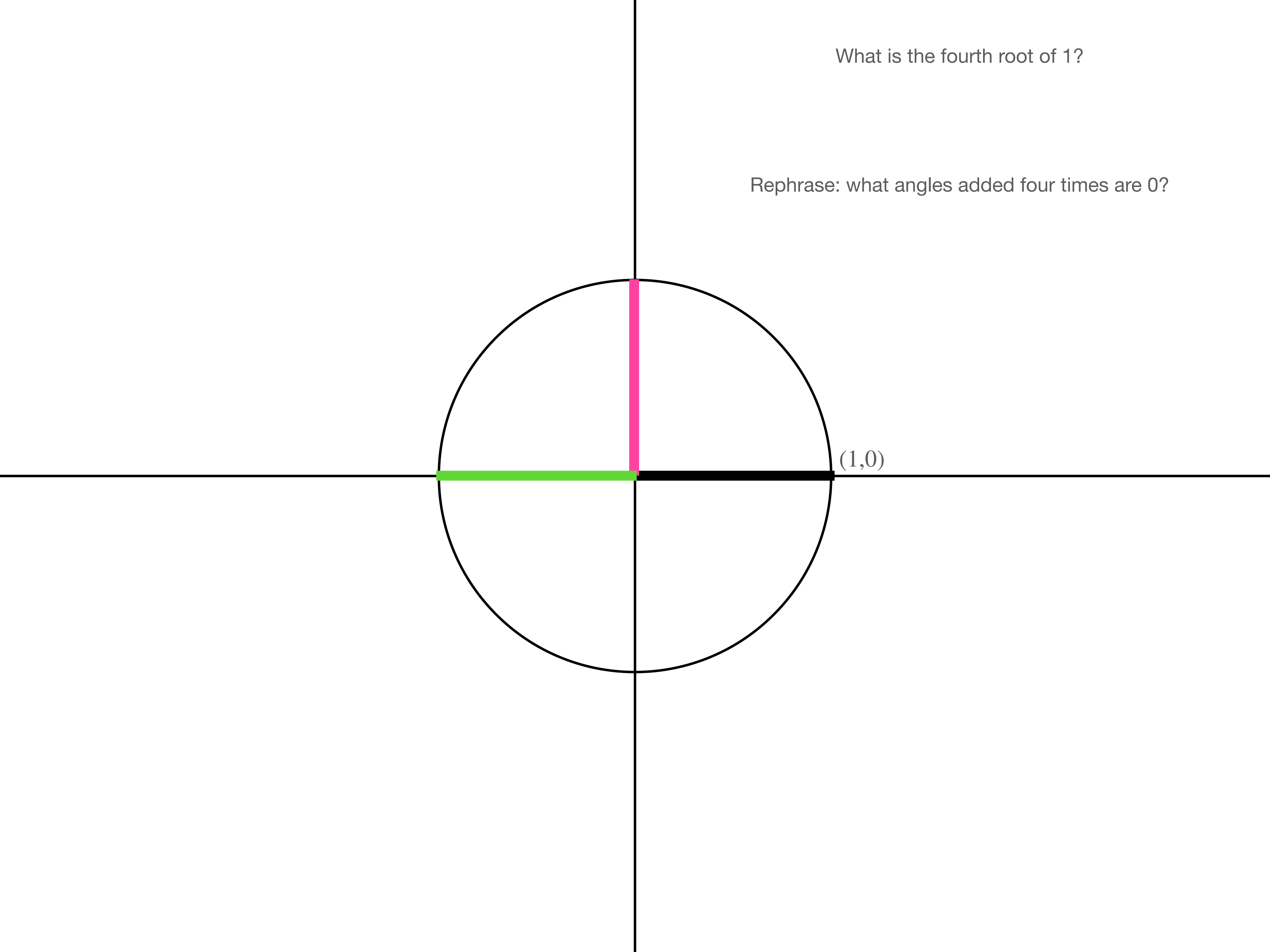

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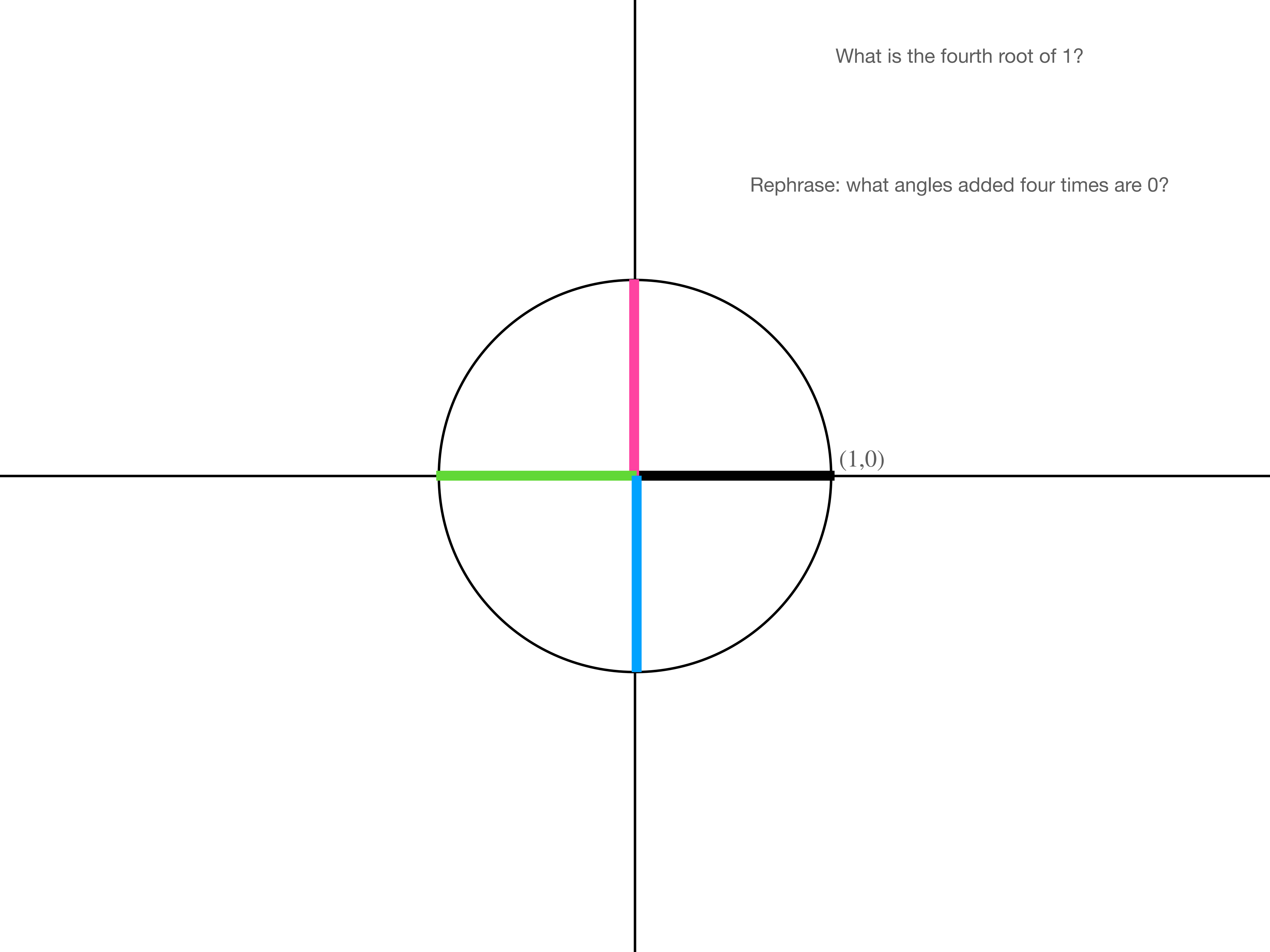
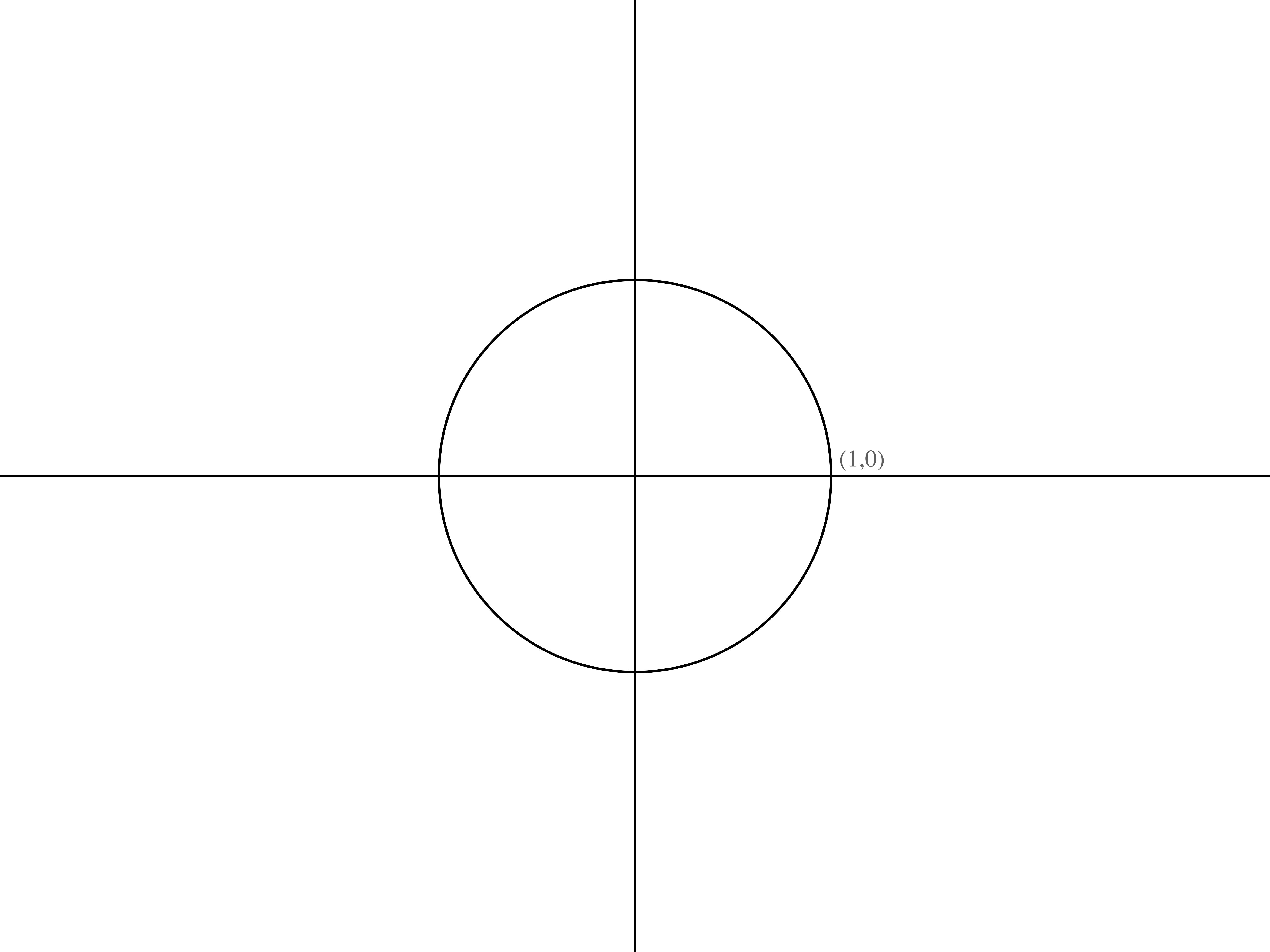
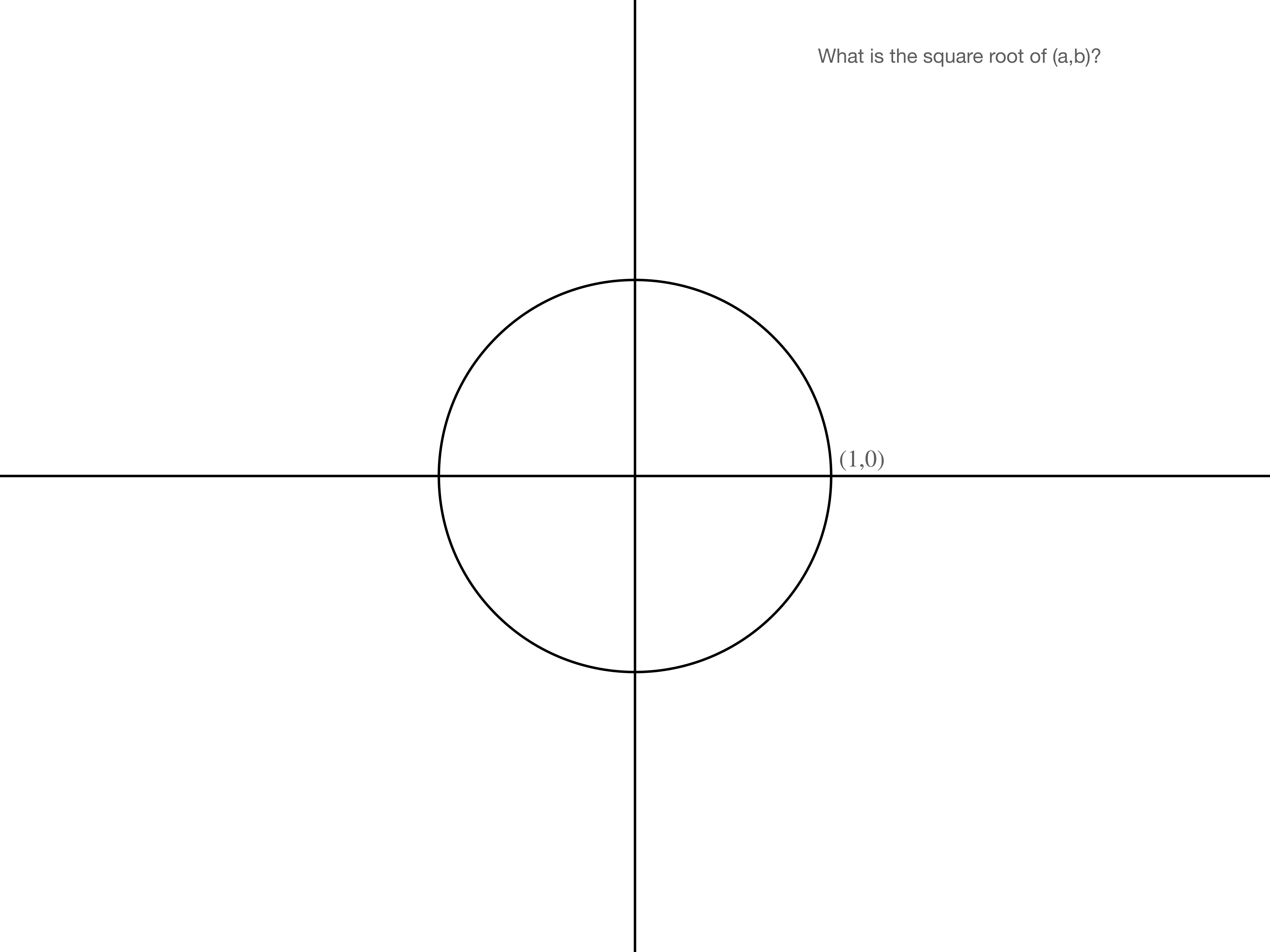


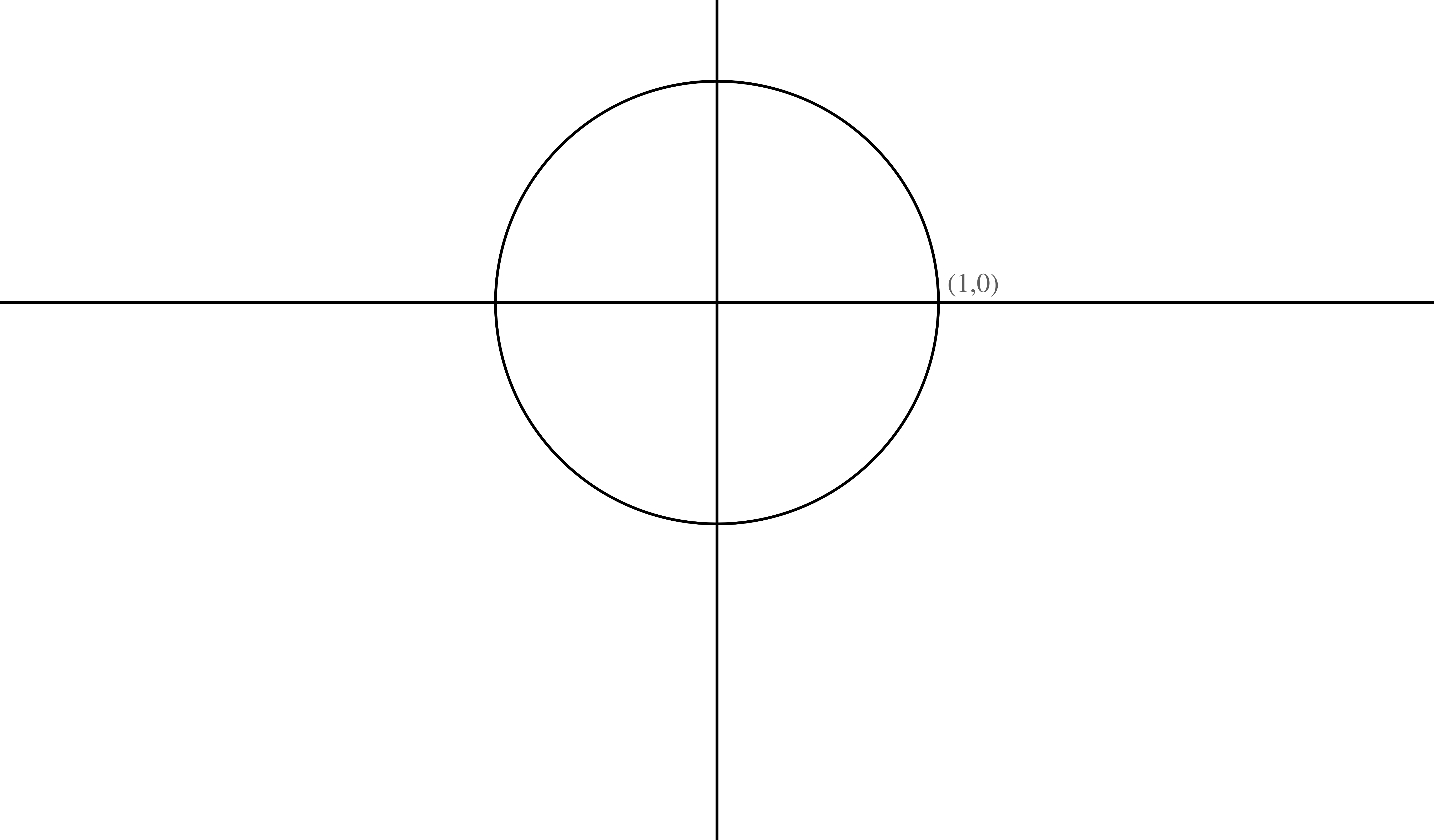

(1,0)

What is the square root of (a,b)?



What is the square root of (a,b)?

Rephrase: what angles added twice point to (a,b)?



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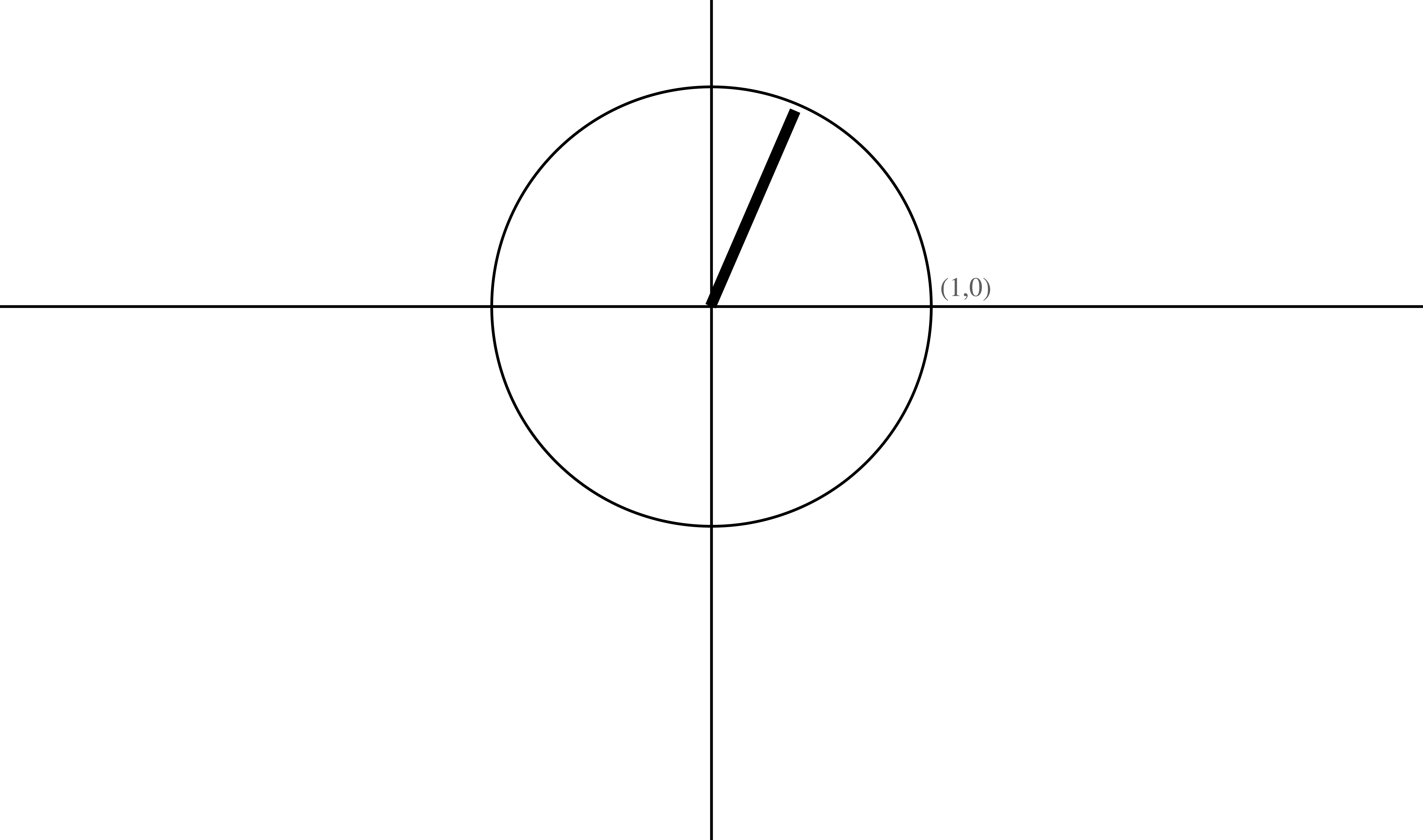

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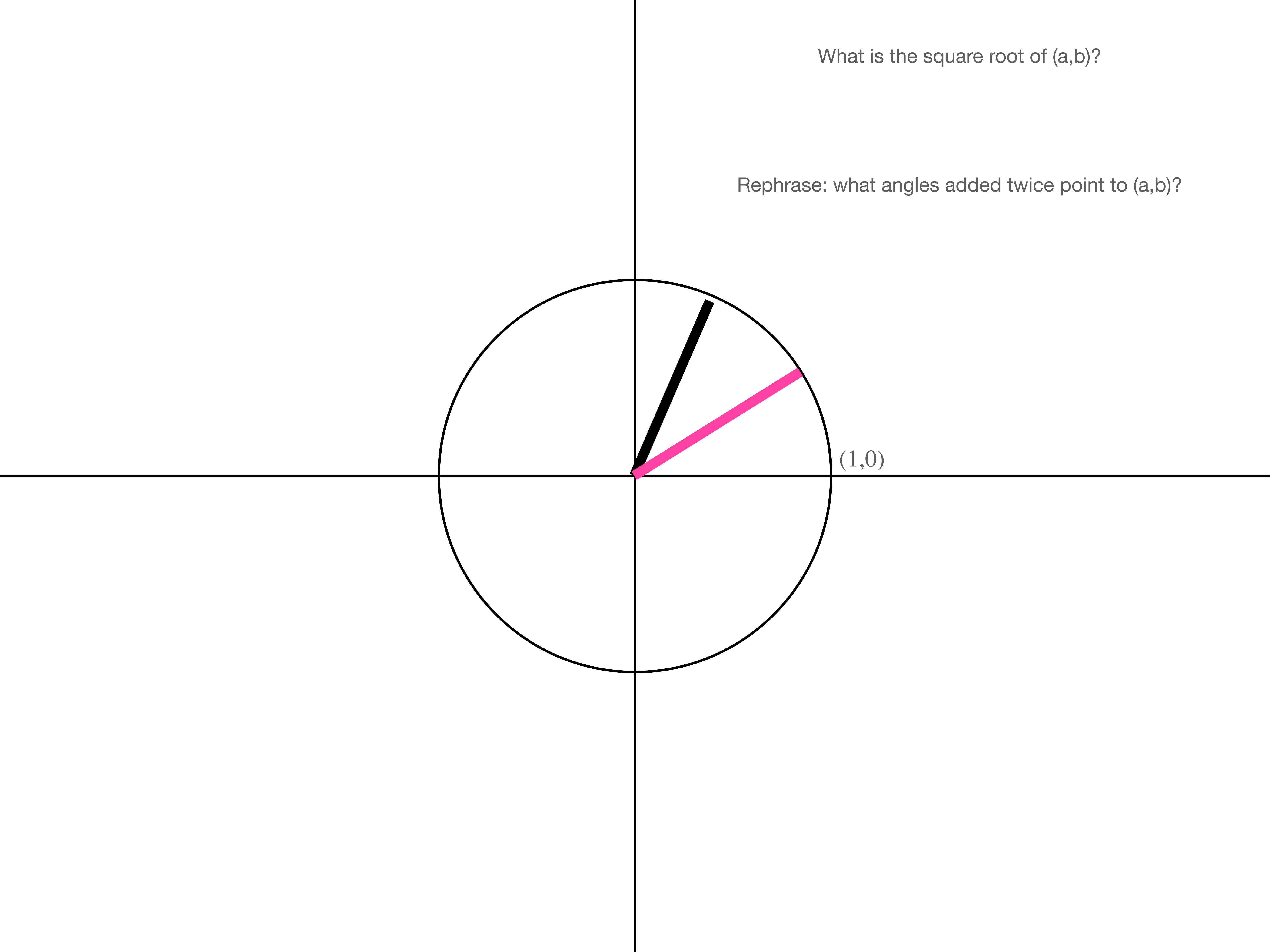
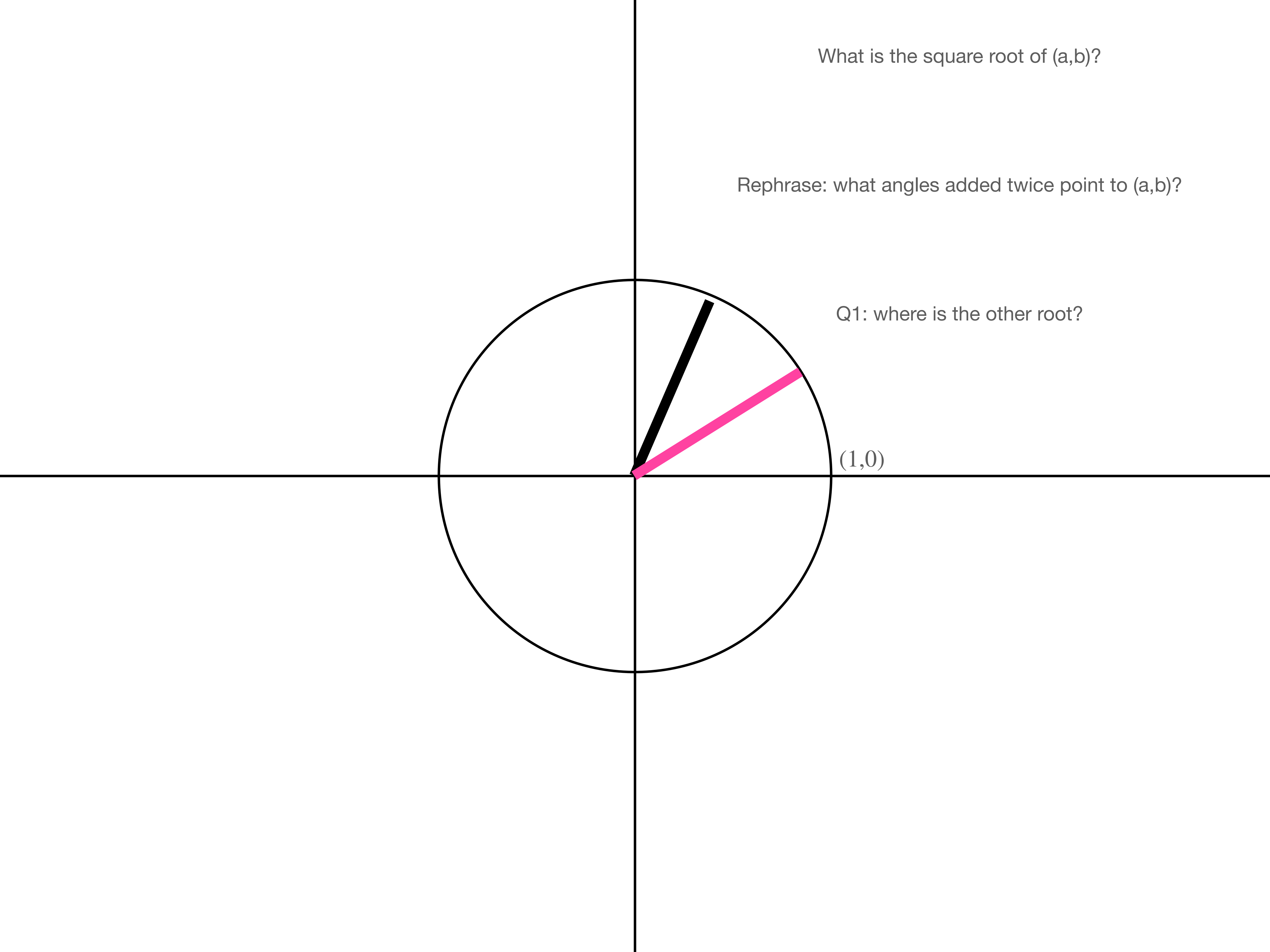
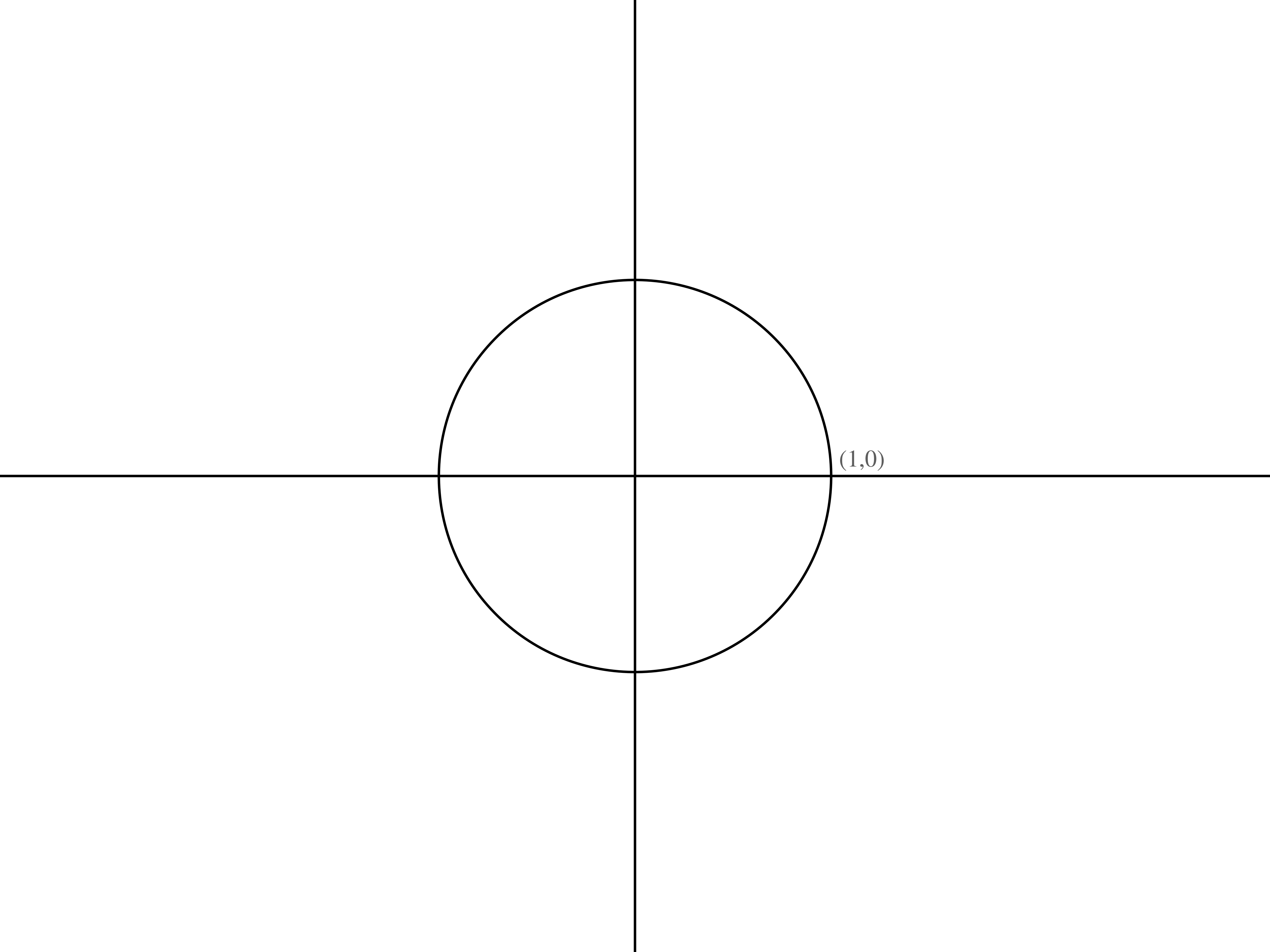


What is the square root of (a,b)?

Rephrase: what angles added twice point to (a,b)?

Q1: where is the other root?

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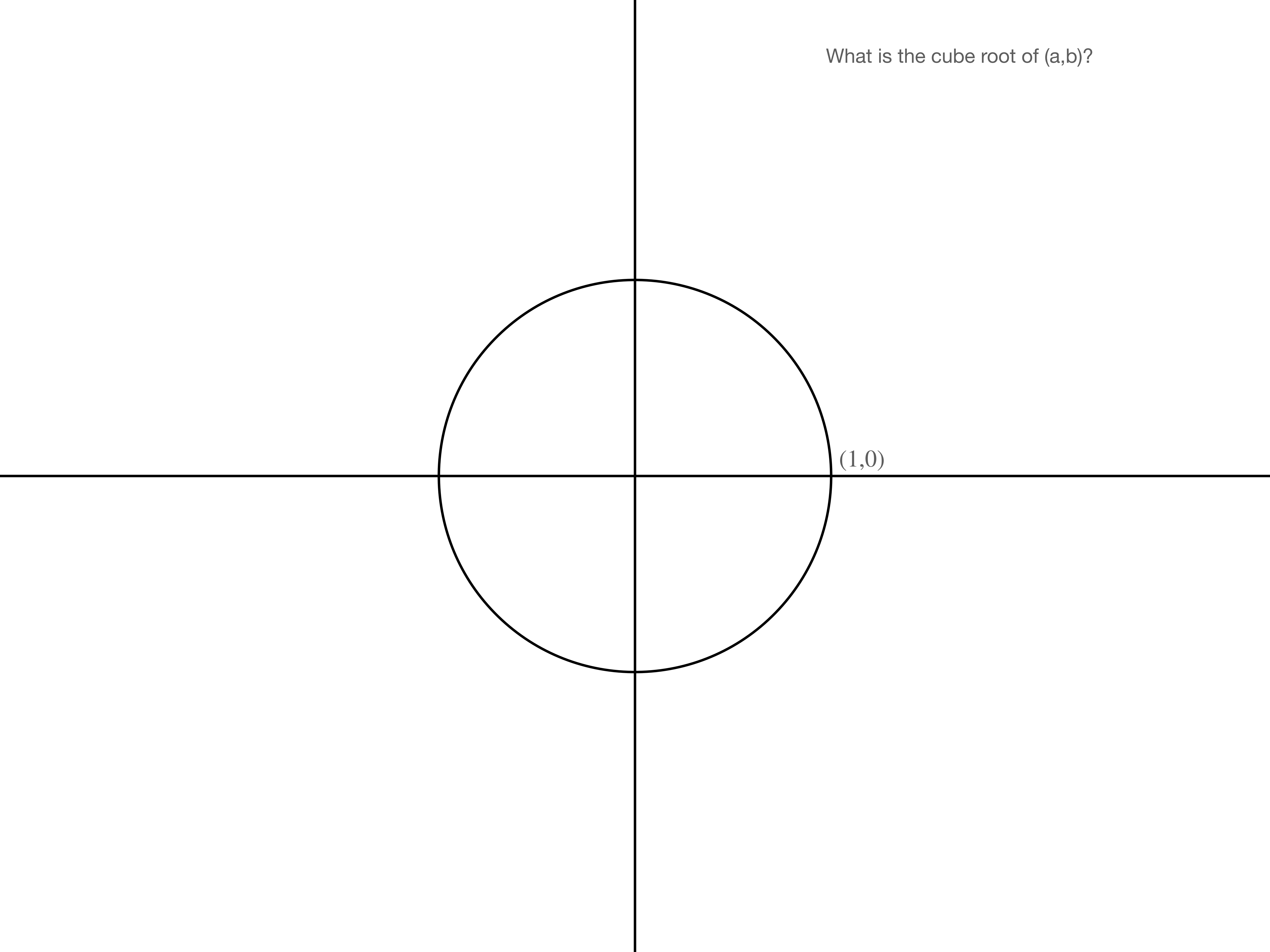
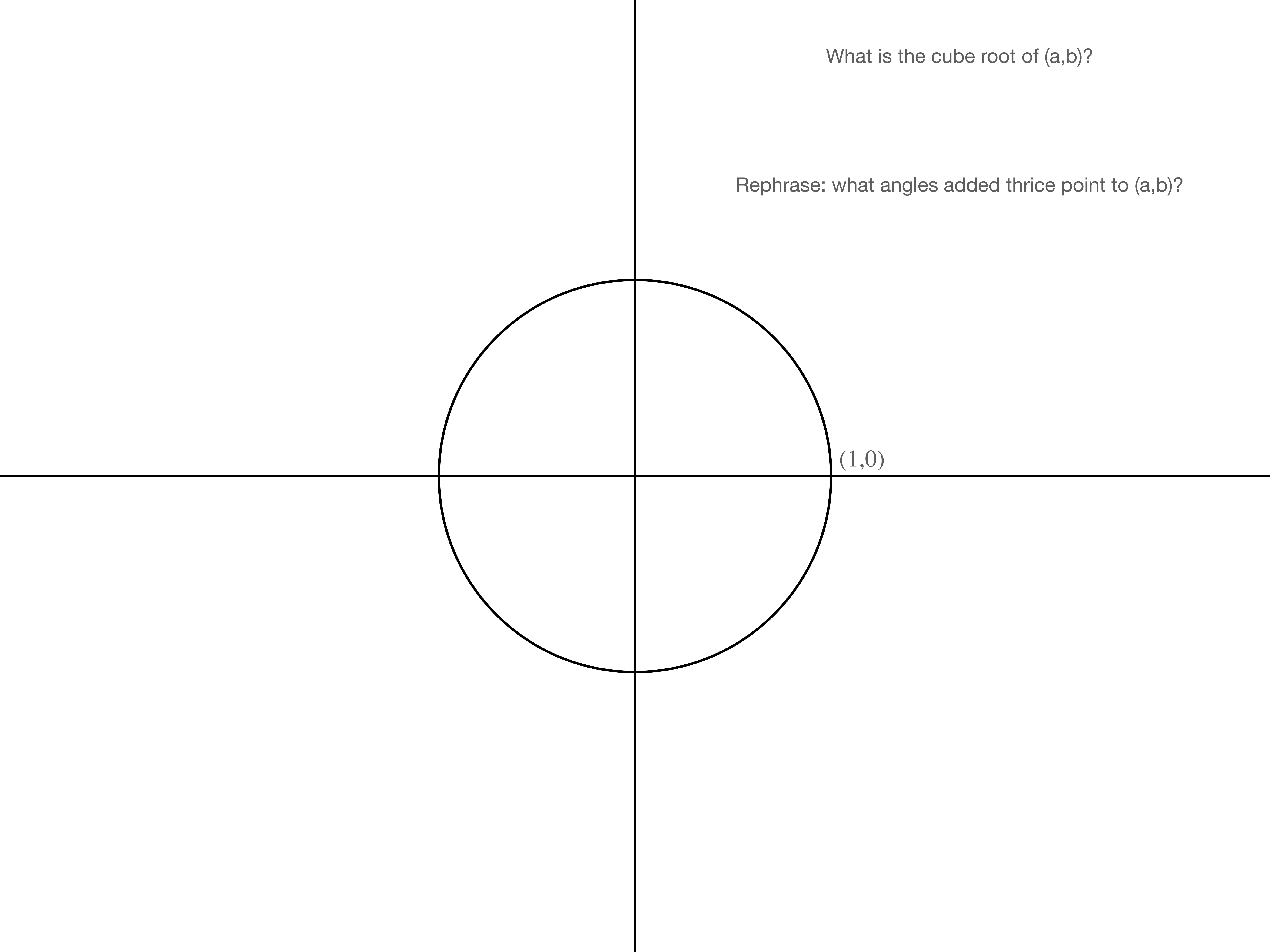


What is the cube root of (a,b)?

Rephrase: what angles added thrice point to (a,b)?



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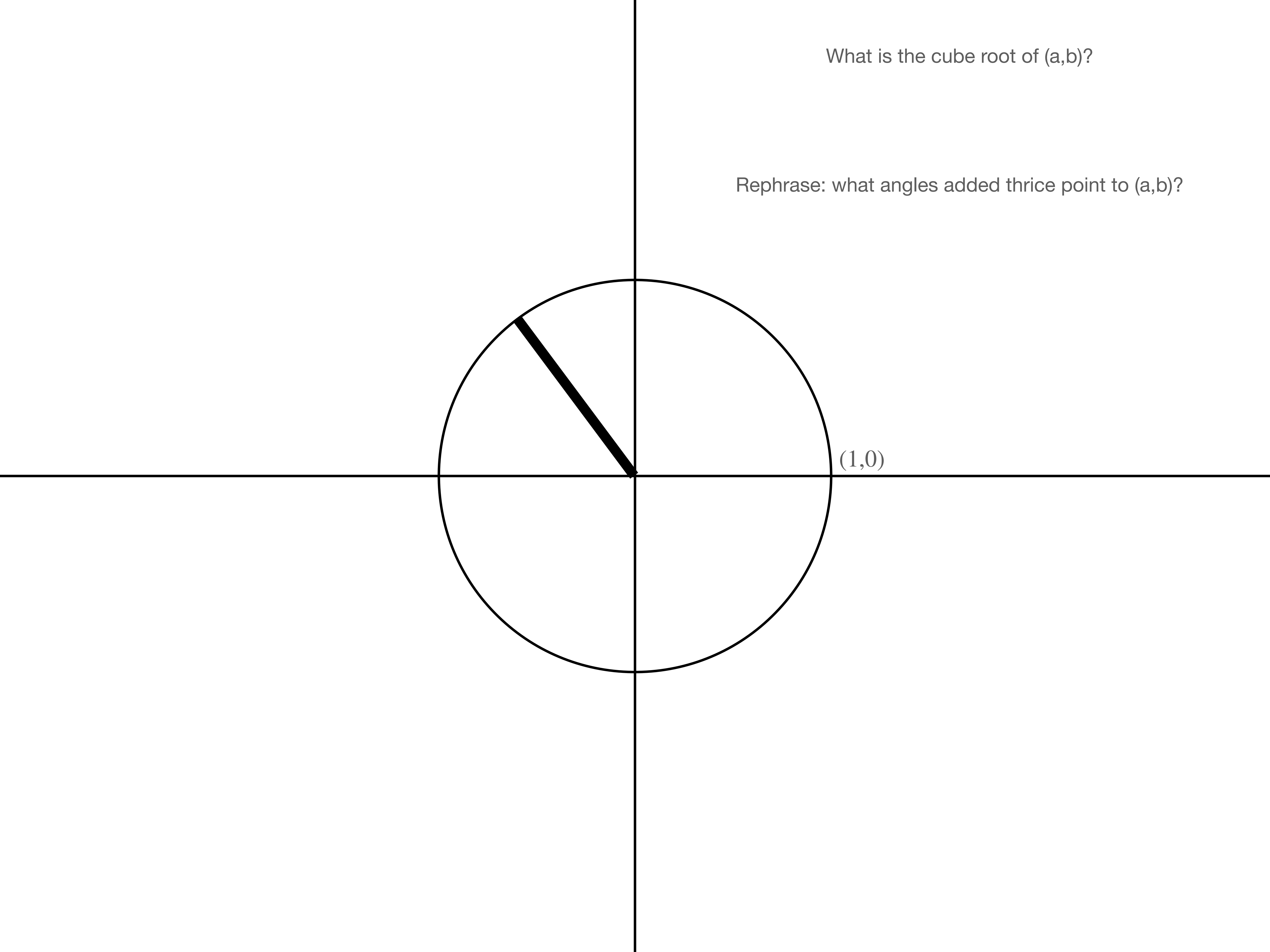

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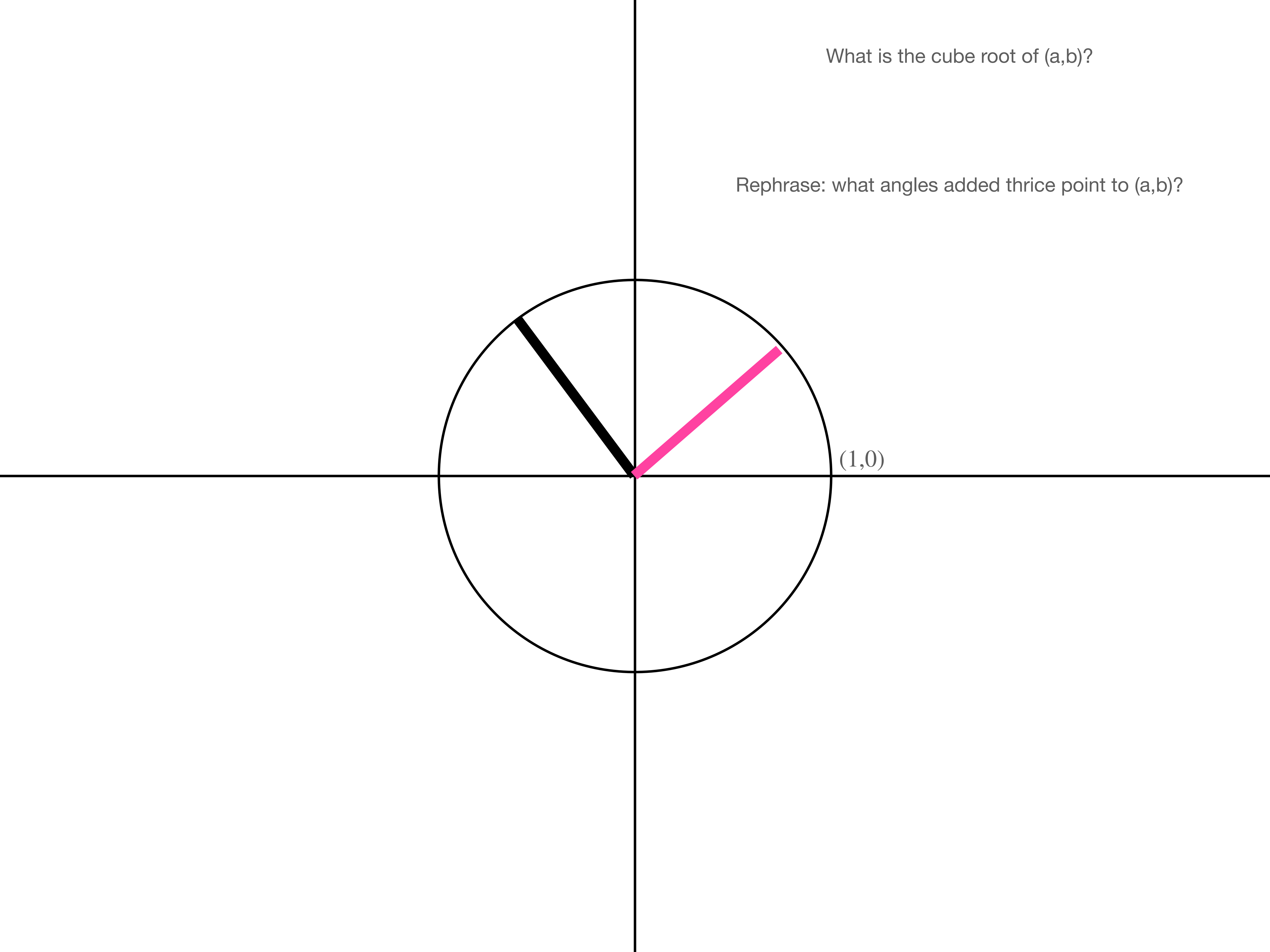

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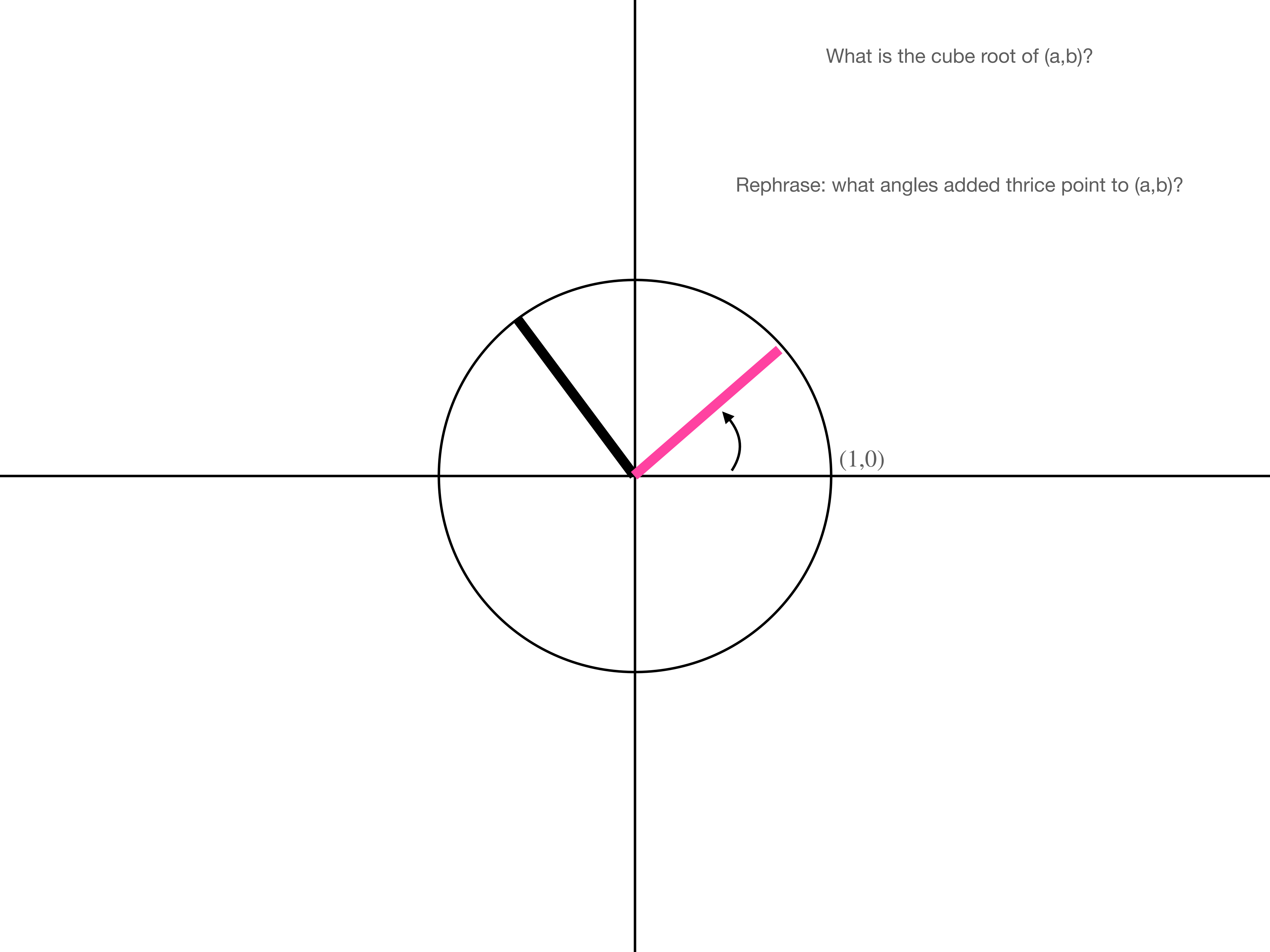

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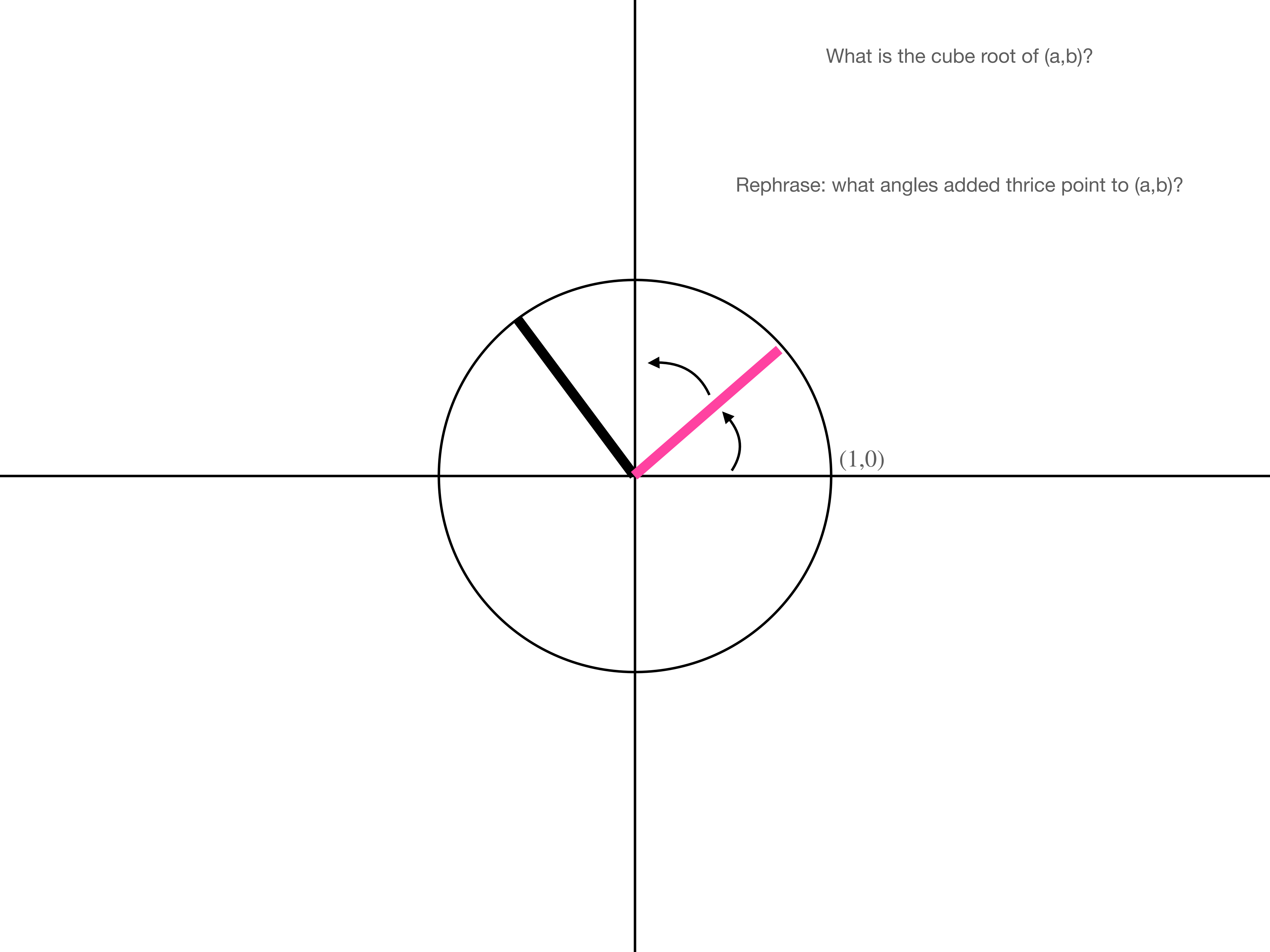

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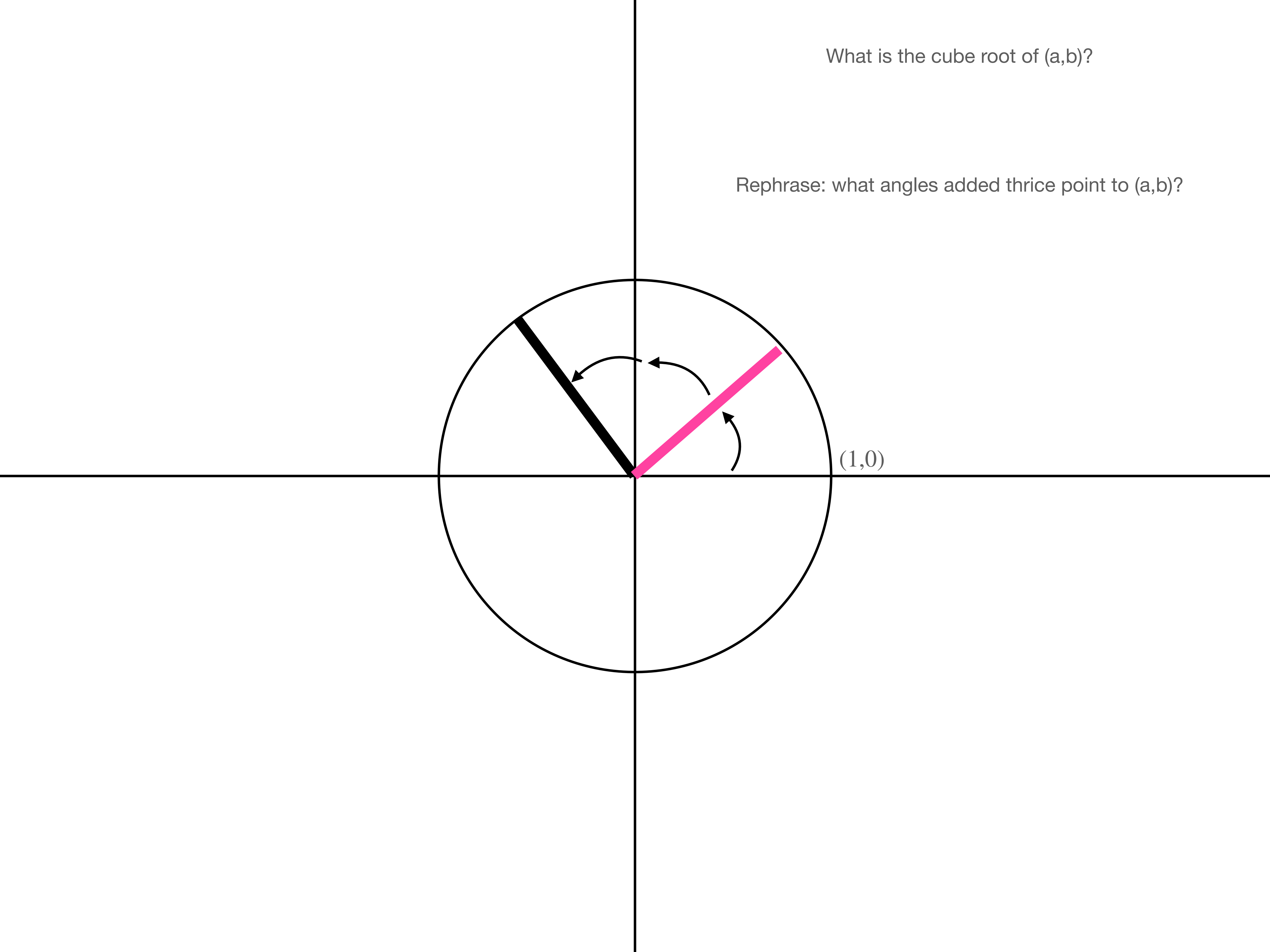
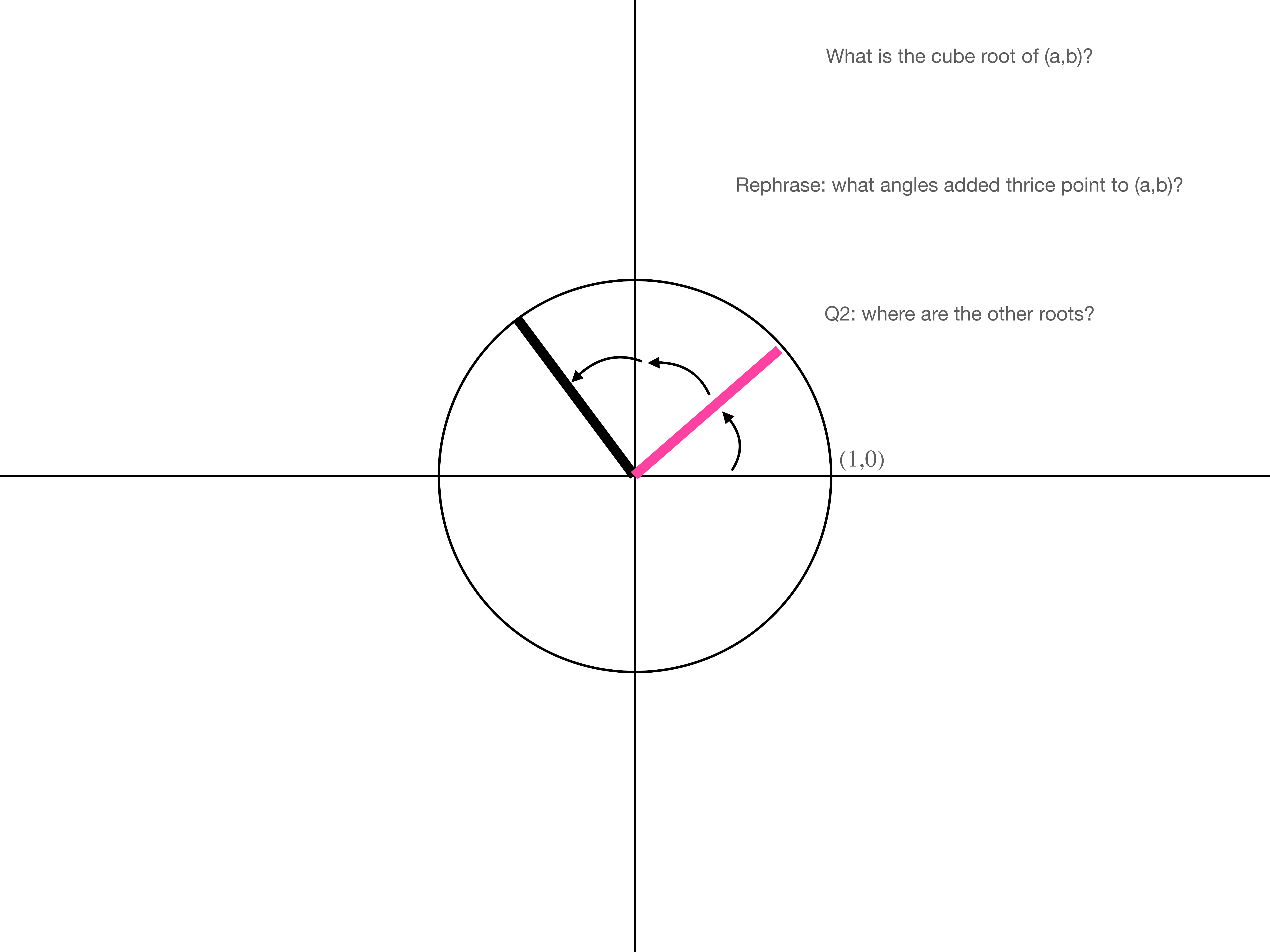
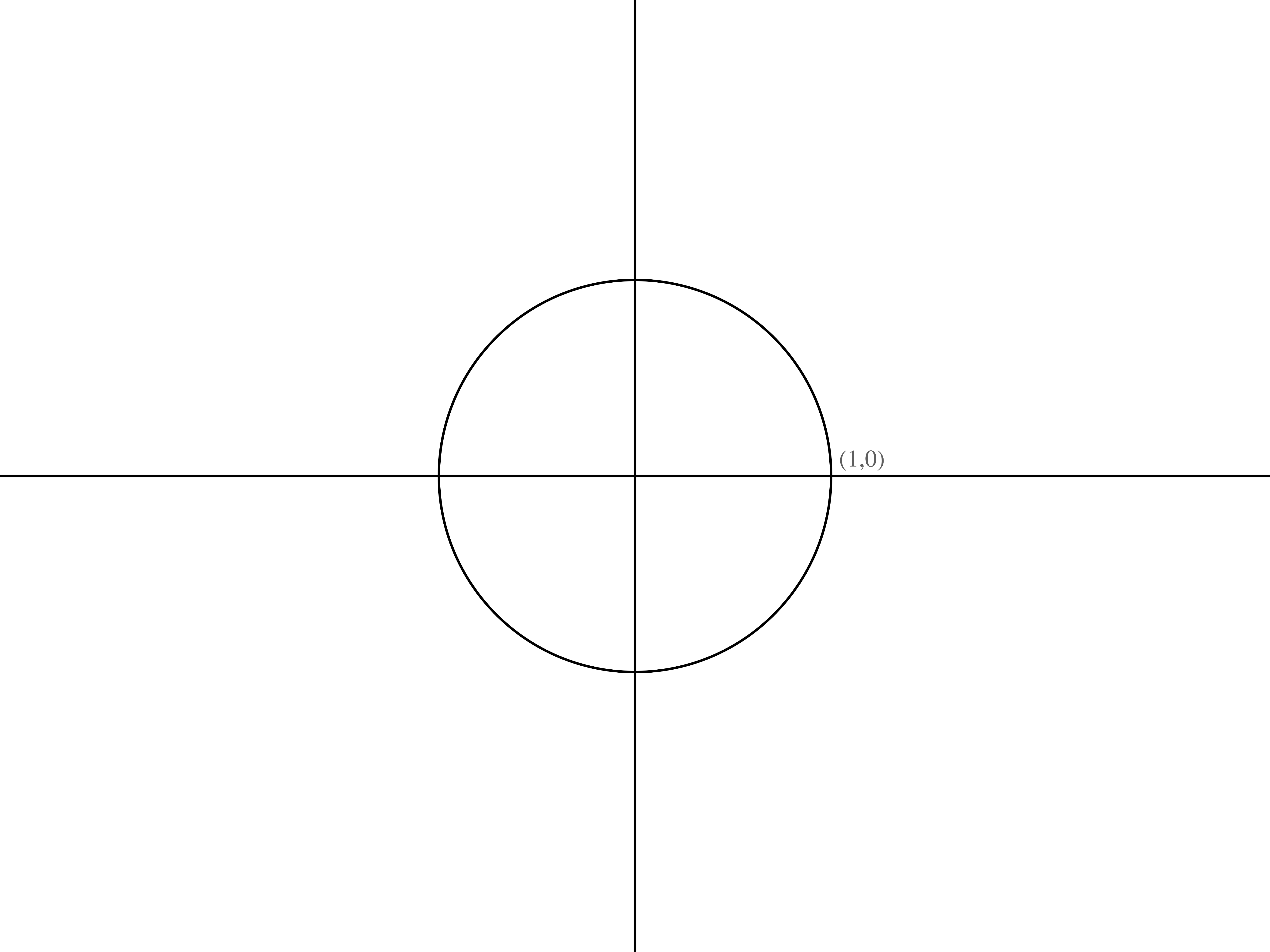


What is the cube root of (a,b)?

Rephrase: what angles added thrice point to (a,b)?

Q2: where are the other roots?

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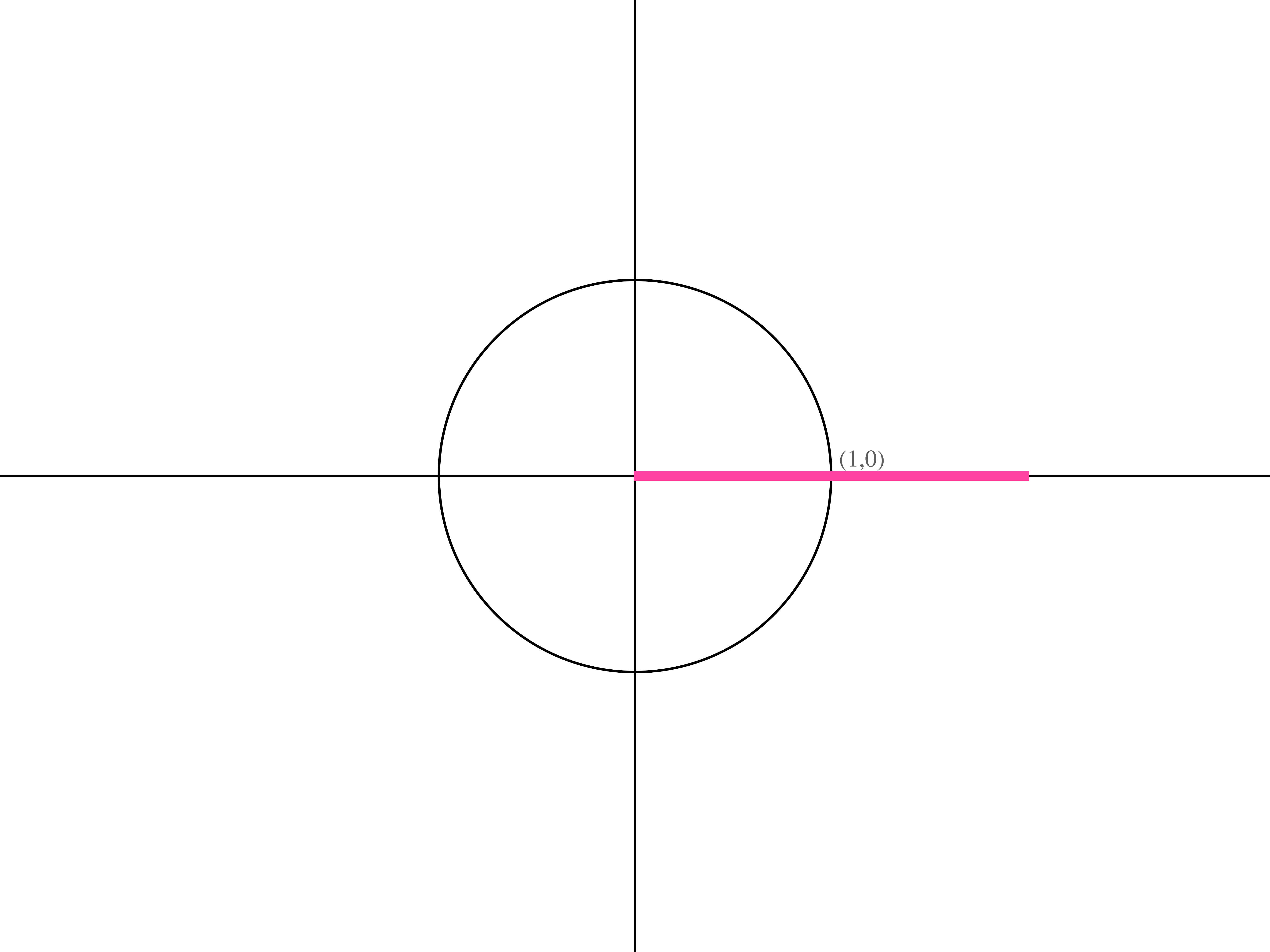
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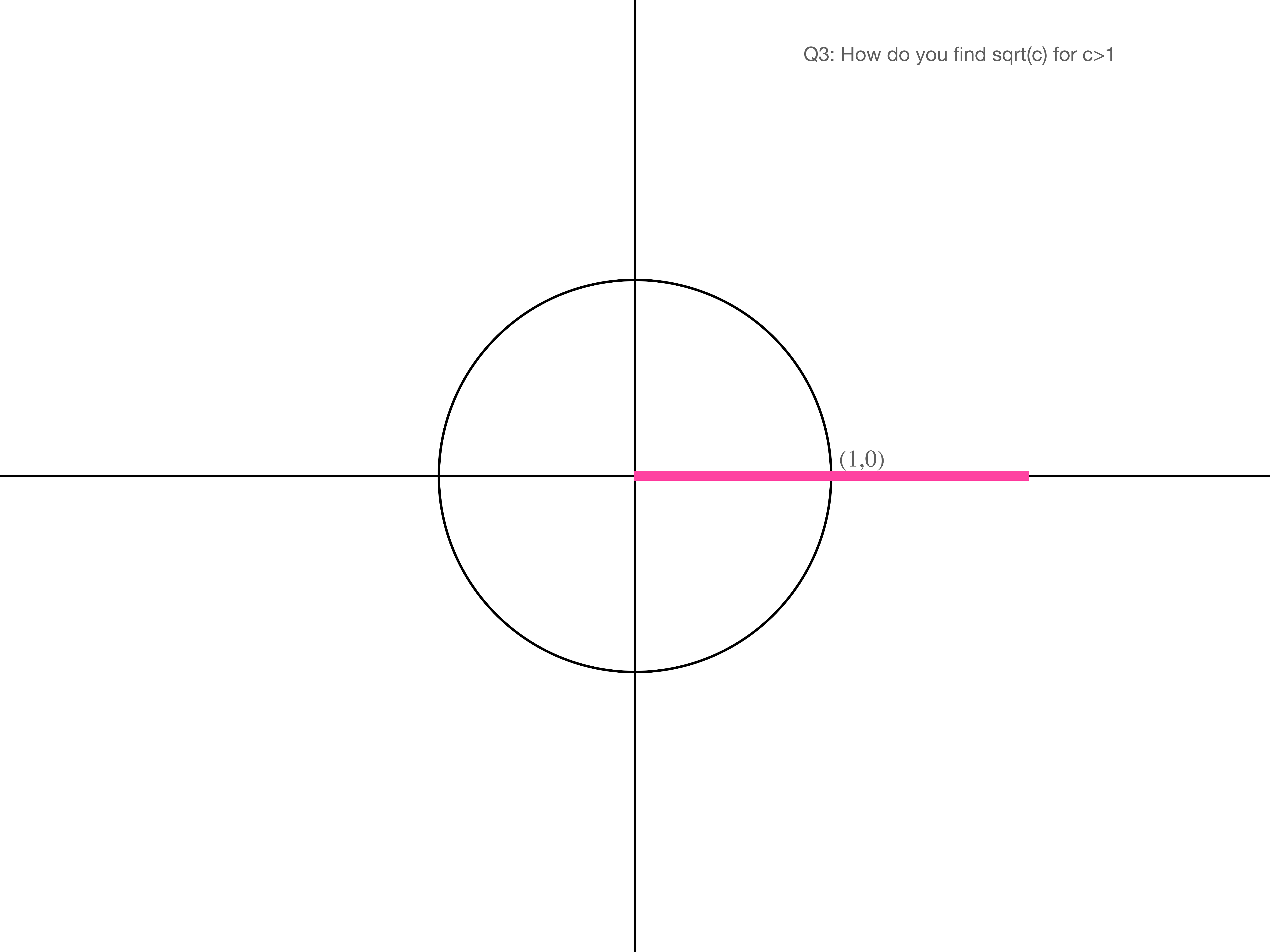
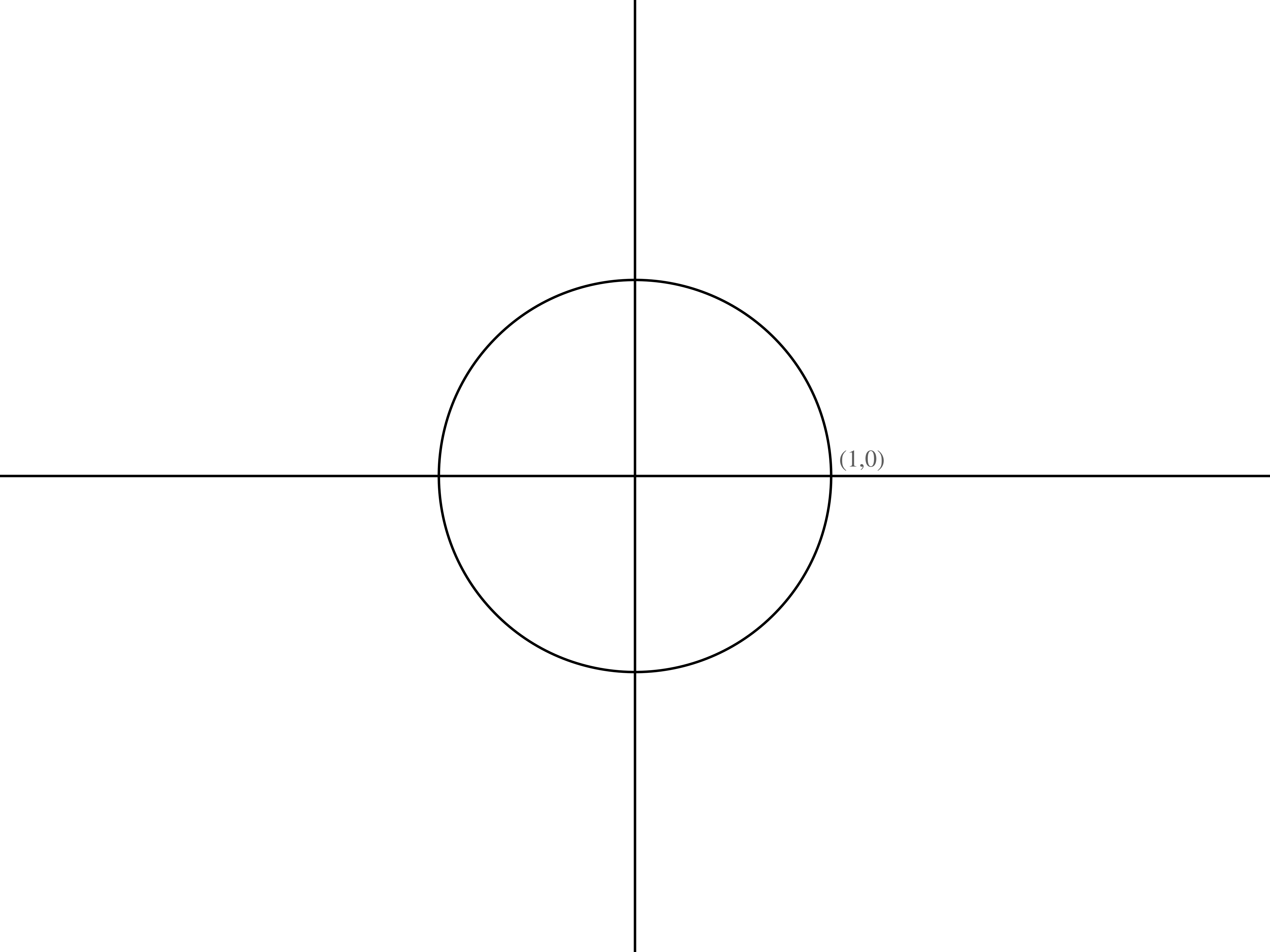
Q3: How do you find sqrt(c) for c>1

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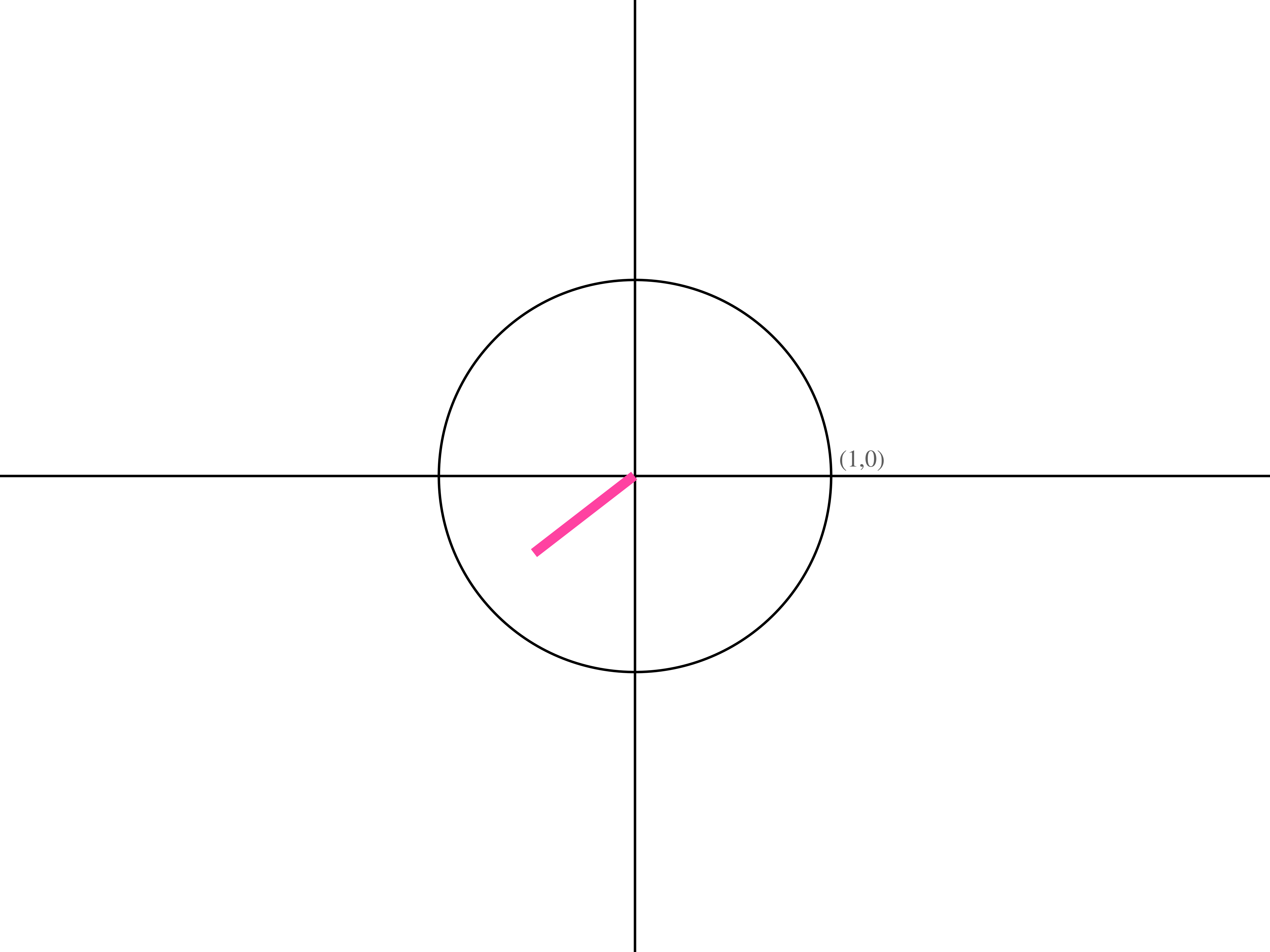
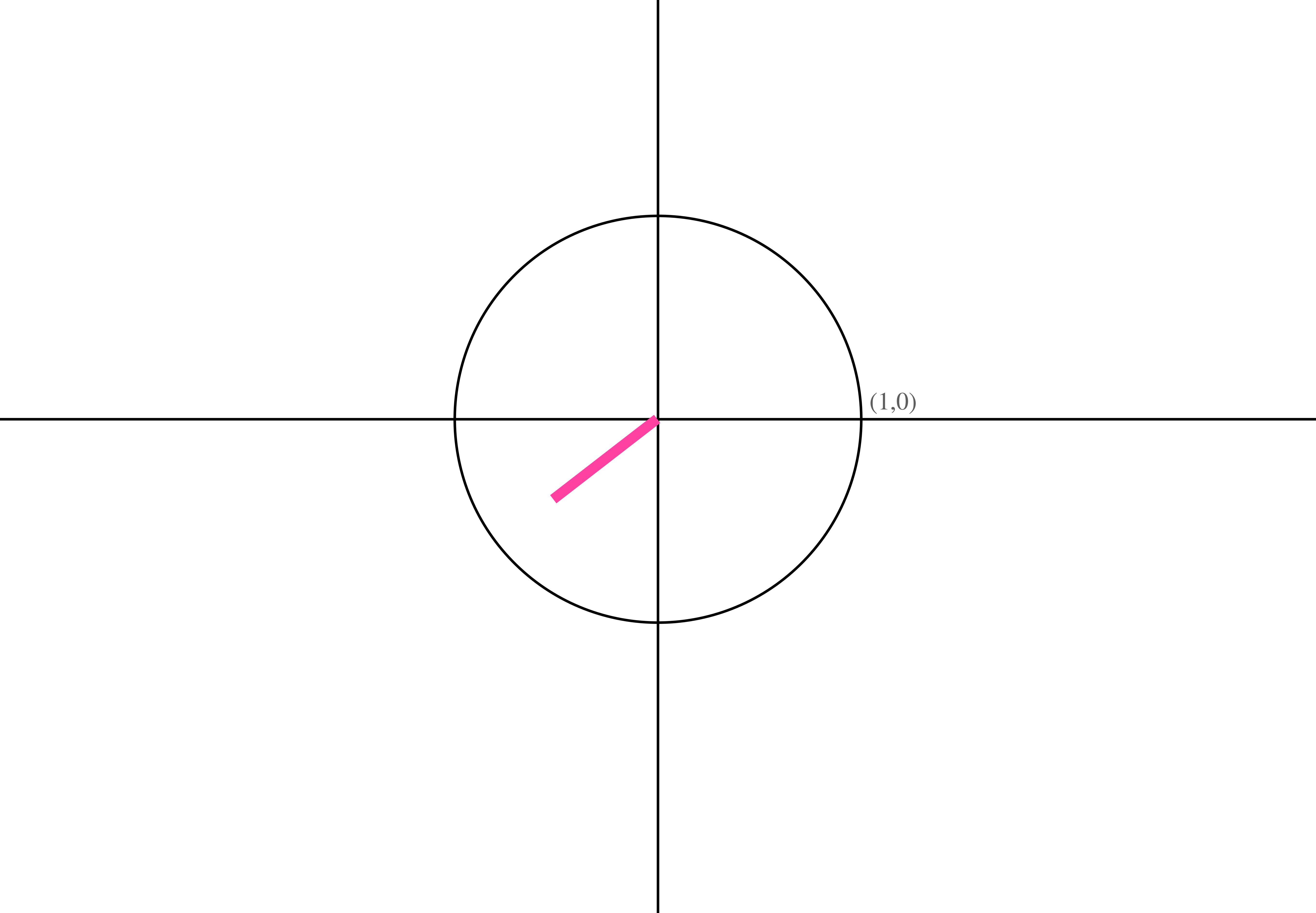
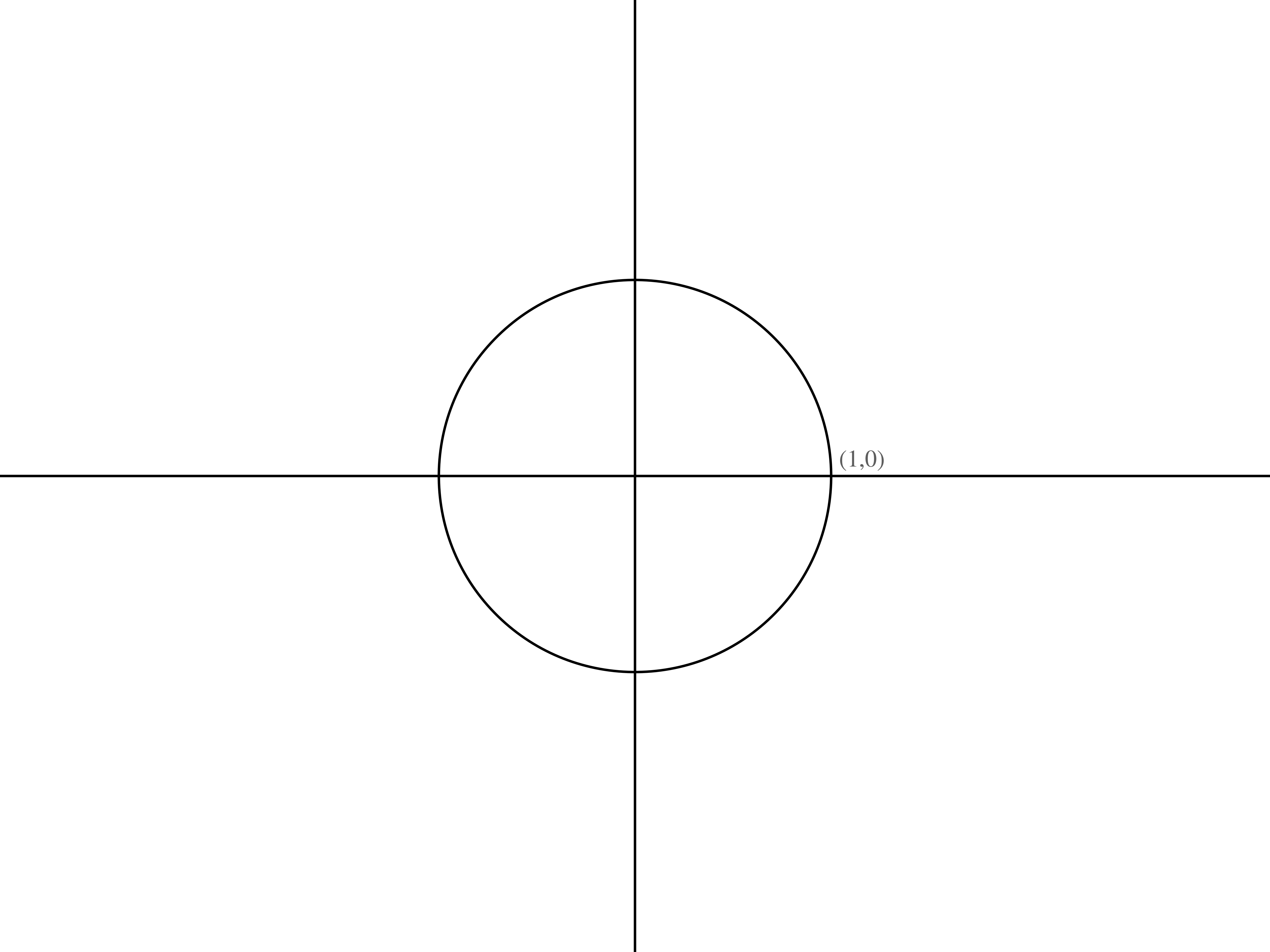
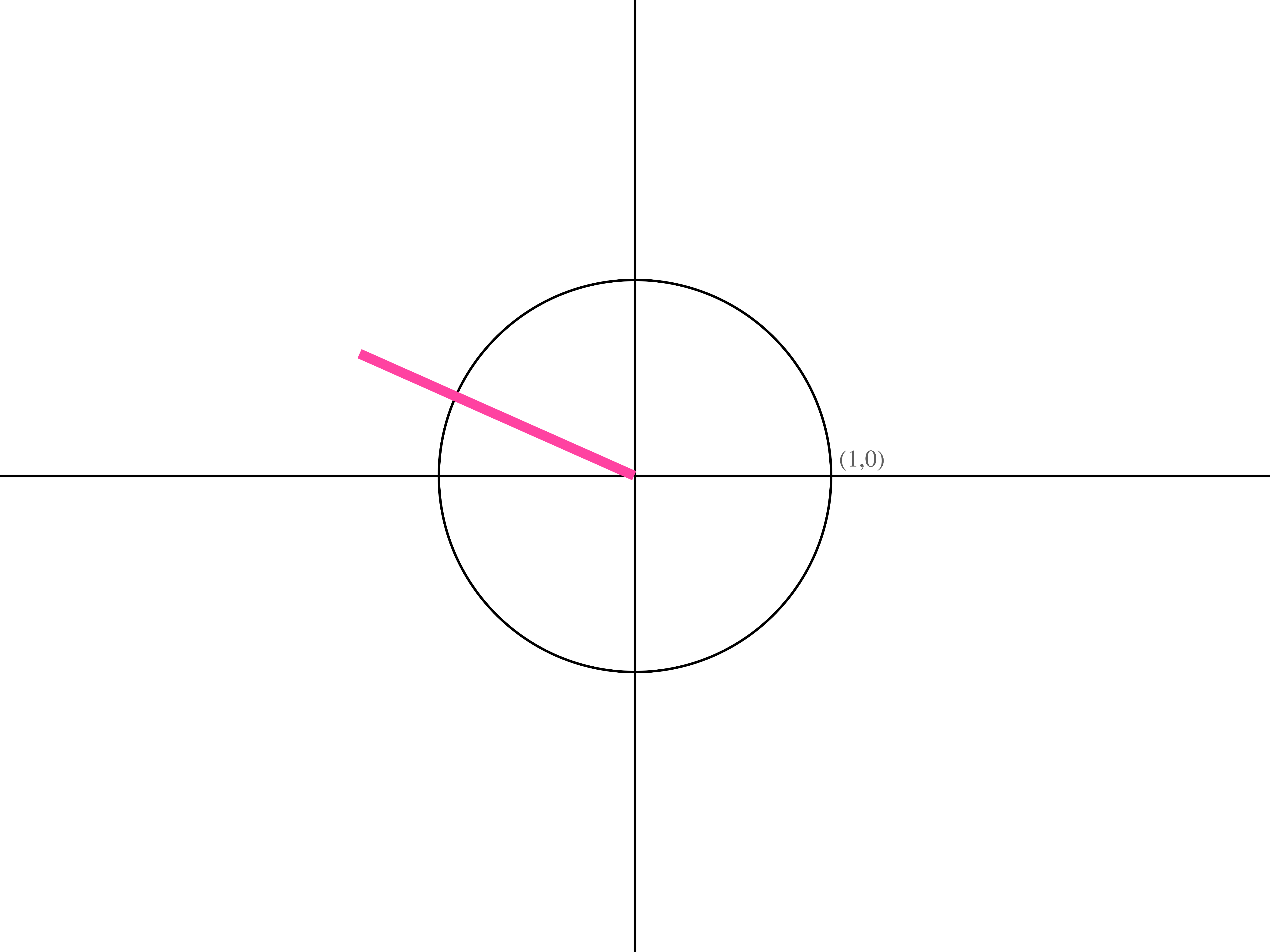

(1,0)

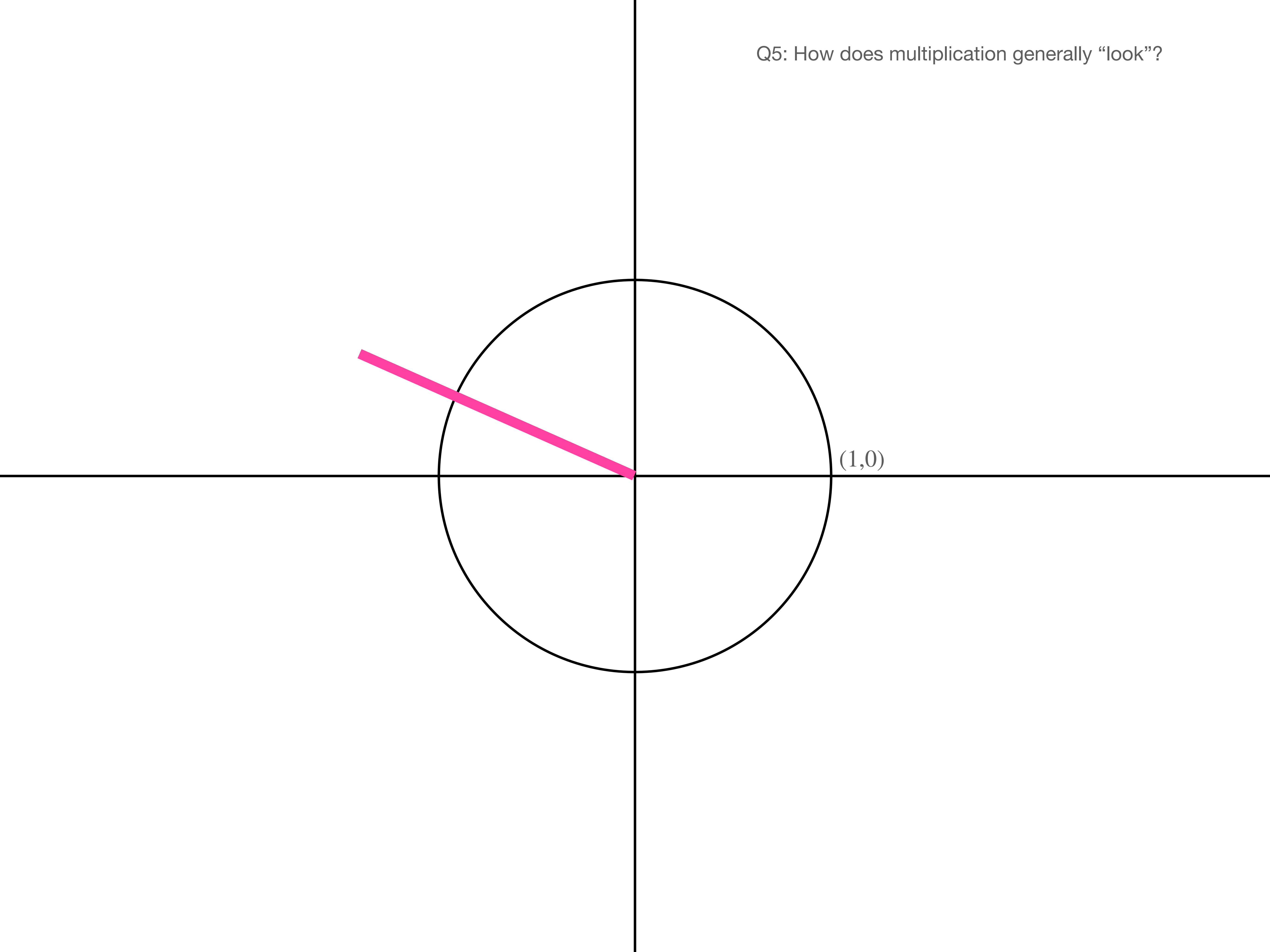
Q4: How do you find sqrt(a,b) in general?




(1,0)


(1,0)

Q5: How does multiplication generally "look"?



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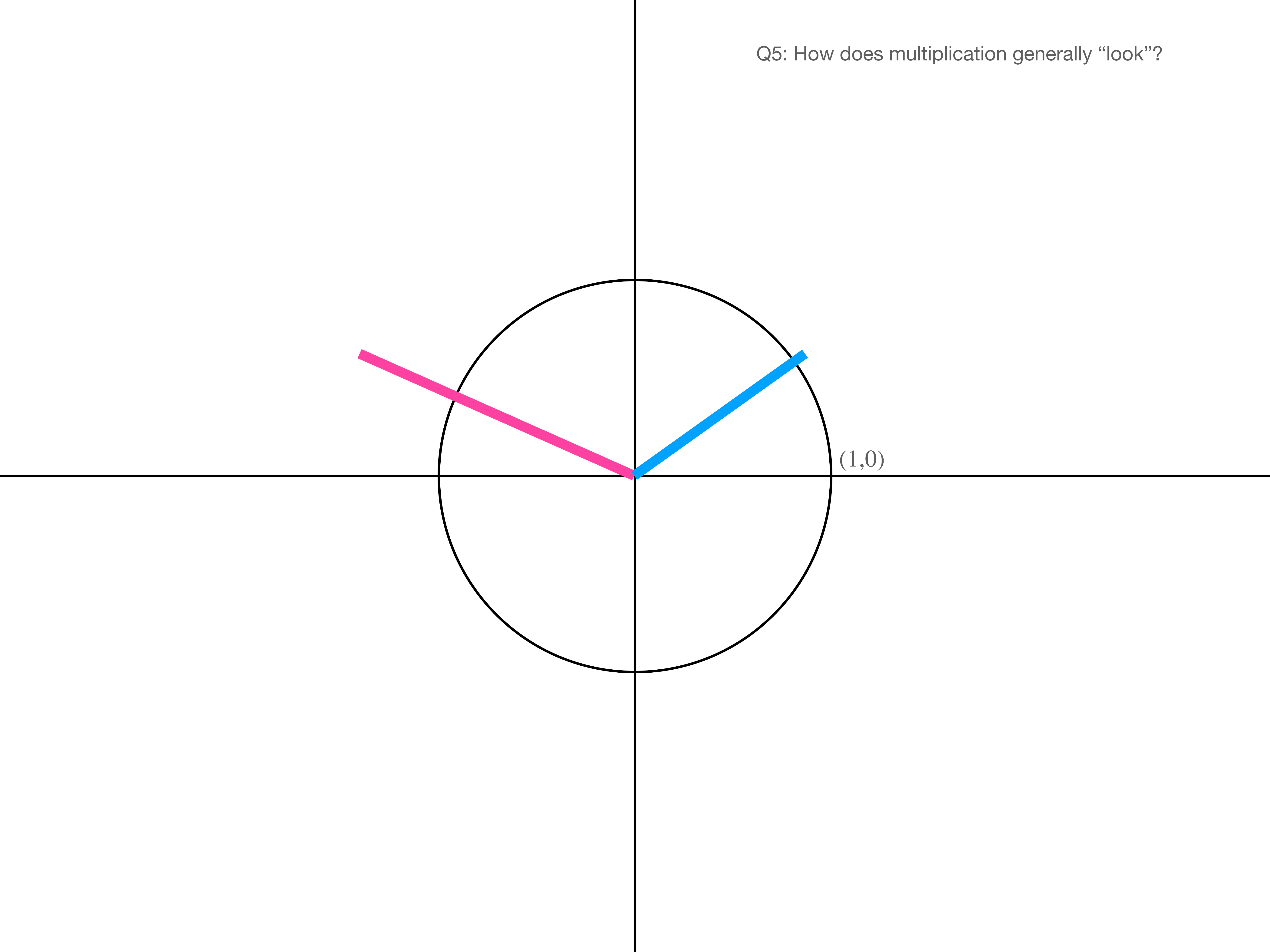

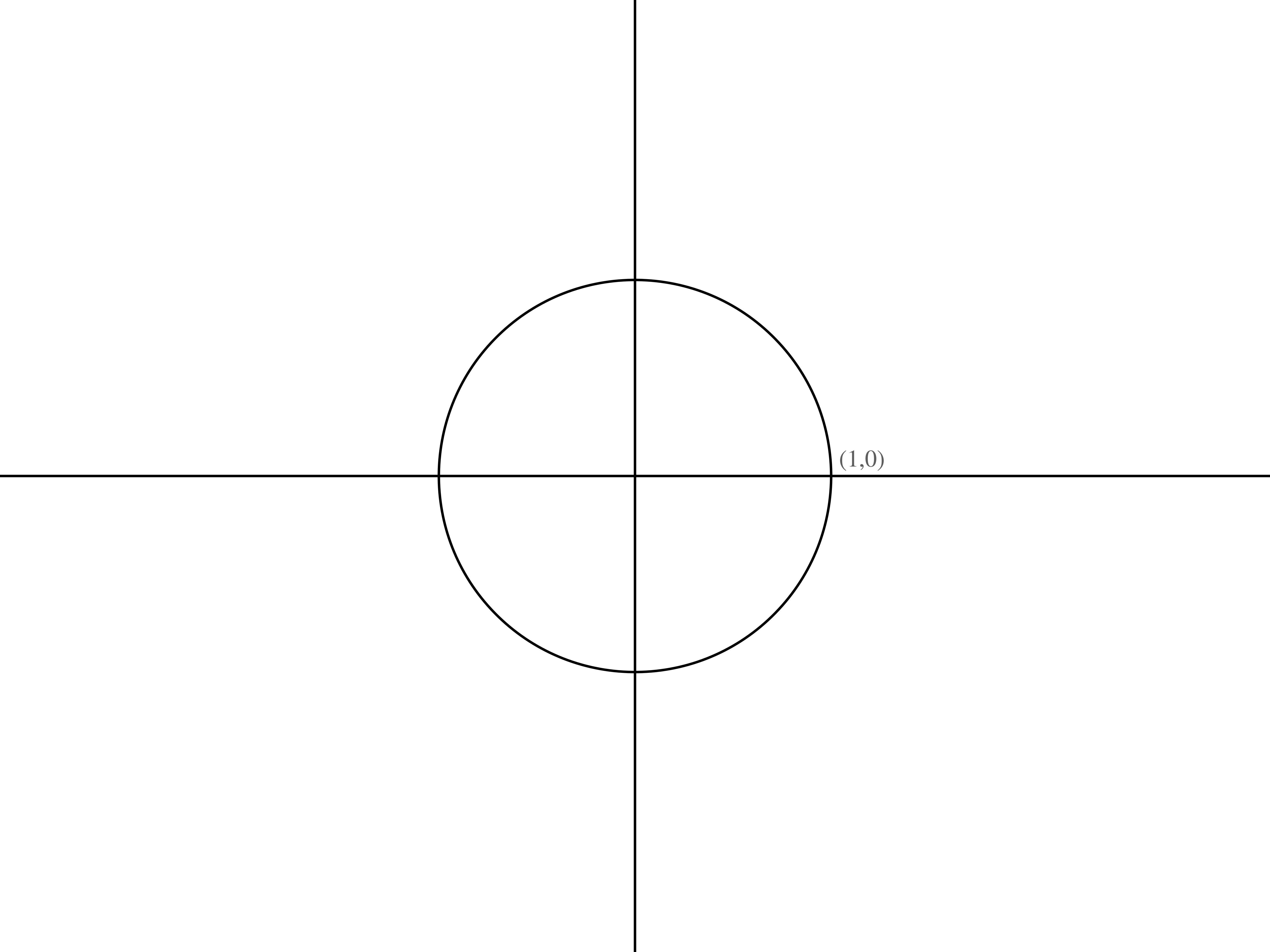
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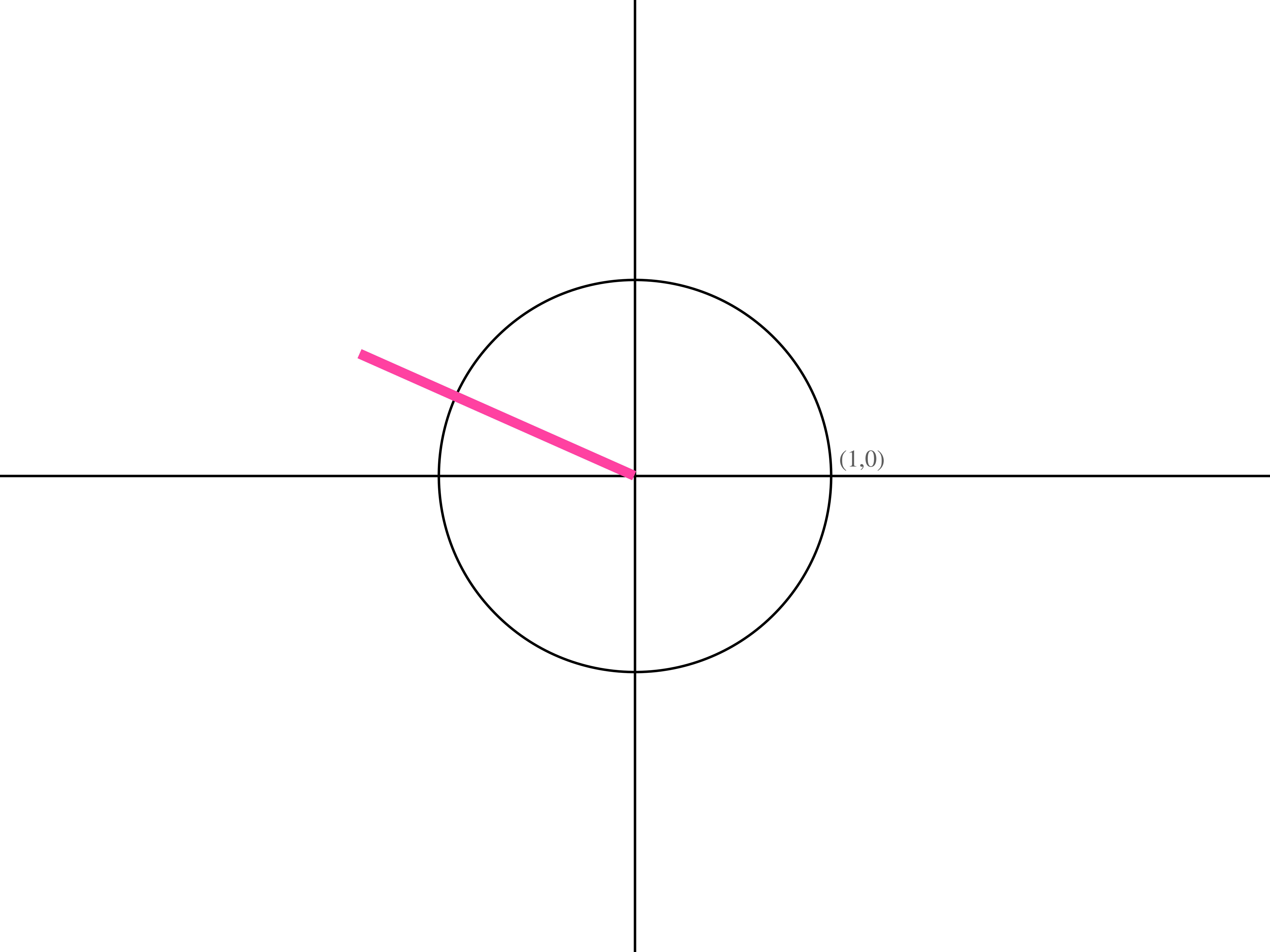
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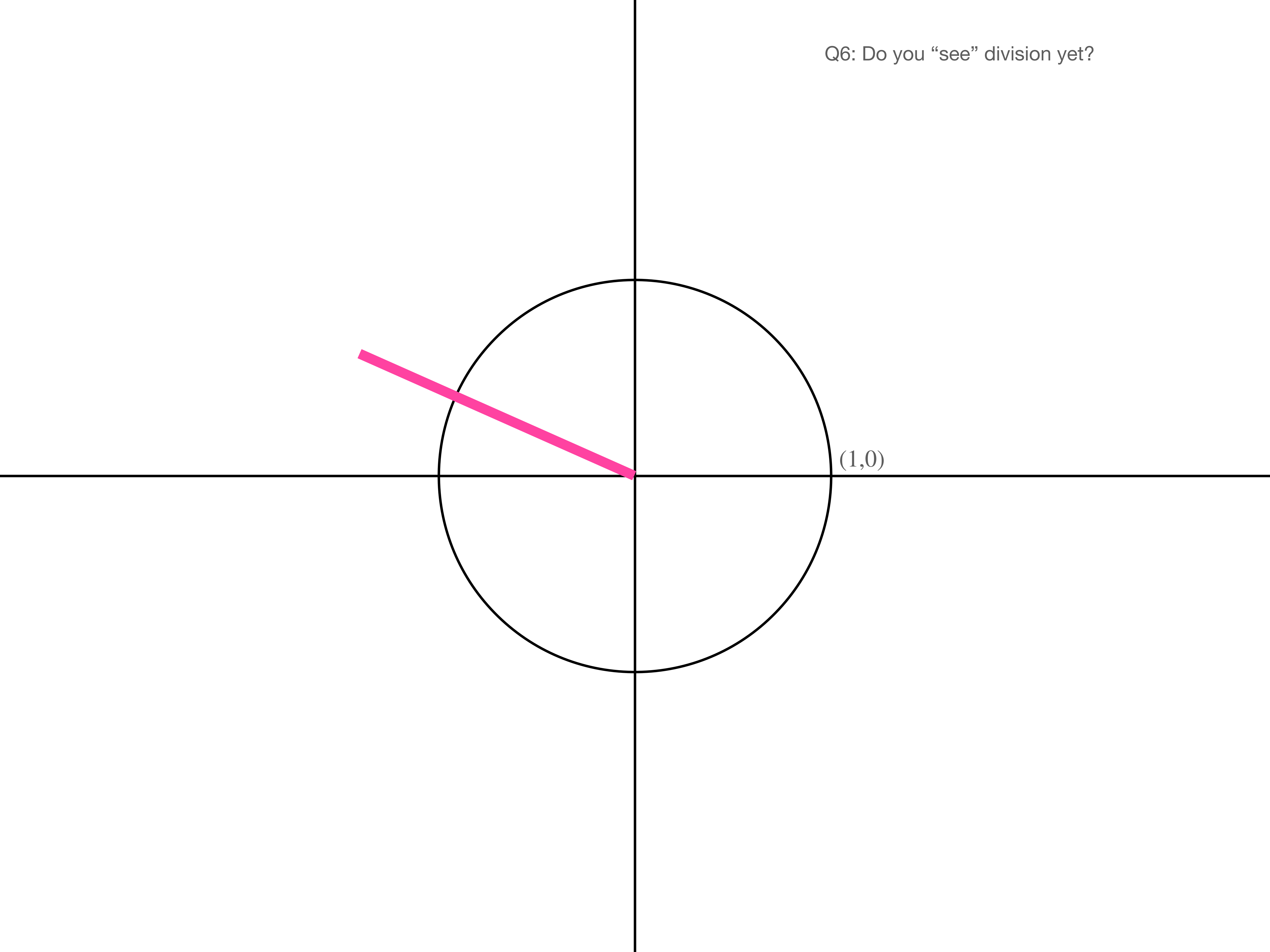
Q6: Do you "see" division yet?



Q6: Do you “see” division yet?